Intellectual Property Financing, Innovation, and Aggregate TFP*

Zhiyuan Chen† Guanliang Hu‡

FEBRUARY 25, 2026

Abstract

We study how expanding access to intellectual property (IP)-backed credit affects firm innovation and aggregate total factor productivity (TFP). Exploiting China's IP-backed financing pilot as a quasi-natural experiment, we use difference-in-differences estimates to show that the policy increases R&D participation, raises firm-level productivity, and reduces capital misallocation. To interpret these effects and quantify their aggregate implications, we develop a dynamic model of heterogeneous firms that borrow subject to collateral constraints backed by tangible assets and IP, and choose R&D investment and the accumulation of wealth. R&D raises future productivity and creates collateralizable IP. We estimate the model by the method of simulated moments, using the policy-induced shift in R&D participation to identify the change in IP pledgeability. The estimated model implies that a nationwide adoption of IP-backed financing would raise long-run TFP by 14%, with roughly two-thirds coming from static gains due to improved capital allocation and the remainder from dynamic gains driven by higher innovation that accumulate over time. Relative to a comparable expansion of lending backed by tangible collateral, IP-backed financing delivers larger productivity gains by easing constraints for high-productivity but low-wealth firms and by strengthening incentives to accumulate IP as collateral for future borrowing.

Keywords: Intellectual property, collateral, financial constraints, R&D investment, TFP

*We thank Treb Allen, Francisco Buera, Lorenzo Caliendo, Angus Chu, Russell Cooper, Jingting Fan, Chao He, Luigi Paciello, Bernardo Ribeiro, and James Tybout for their helpful comments and suggestions. We thank various seminar participants at CUEB (ISEM), CUHK-Shenzhen (SEM), Liaoning University, UIBE and University of Macau (FSS-DECO). Zhiyuan Chen is indebted to the supervision by Jonathan Eaton and Mark Roberts.

†Business School, Renmin University of China; Email: chenzhiyuan@rmbs.ruc.edu.cn

‡Department of Economics and Finance, City University of Hong Kong; Email: guanlihu@cityu.edu.hk

1 Introduction

The modern economy is increasingly driven by intangible assets which now constitute a substantial portion of firm value, particularly for innovative enterprises (**???**). Despite this shift, traditional lending practices remain heavily biased toward tangible collateral, creating a mismatch between the asset composition of modern firms and the requirements of financial markets. This excludes the possibility of using valuable intangible assets as collateral and borrowing from credit markets, potentially hampering innovation-driven economic growth. Recognizing this challenge, policymakers have begun experimenting with policies that enable firms to leverage their intellectual property (IP hereafter) as collateral (**??**), yet comparatively less is known about the aggregate implications of such policies. This paper studies the aggregate productivity effects of policies expanding access to IP-backed collateral and the underlying mechanisms.

Exploiting China's IP-backed financing pilot program as a quasi-natural experiment, we study how expanded access to IP-backed credit affects firm-level R&D investment and productivity, as well as the across-firm allocative efficiency of capital. To interpret and quantify these effects, we develop and estimate a general equilibrium model in which heterogeneous firms choose capital and R&D subject to collateralized borrowing constraints backed by both tangible assets and IP; R&D raises future productivity while generating collateralizable IP. The model maps policy-induced expansions in intangible collateral into aggregate total factor productivity (TFP) through two channels: within-firm productivity improvements and between-firm reallocation of capital across firms. We estimate the model using the method of simulated moments and decompose the aggregate TFP effects of IP-backed financing into (1) static gains from improved allocation and (2) dynamic gains from higher firm-level productivity. Finally, we benchmark IP-backed financing against alternative policies that expand credit through tangible collateral to assess their relative efficiency in raising aggregate productivity.

Our empirical analysis draws on three primary data sources. The first dataset is from China's "National intellectual property pledge financing pilot work program", containing the pilot cities and their starting years as announced by State Intellectual Property Office of China (SIPO). We merge this dataset with two firm-level datasets, with the Annual Surveys of Industrial Production covering pre-treatment years, and Administrative Income Tax Records covering post-treatment years. Exploiting the initiation of China's pilot program for IP-backed financing as a unique quasi-natural experiment, we employ the Difference-In-Differences (DID hereafter) approach and robustly find that this policy significantly increases firms' R&D investment and productivity, and improves across-firm capital allocation efficiency.

While existing quantitative studies demonstrate that financial constraints reduce aggregate productivity through capital misallocation (**???????**) and suppressing innovation activities or affecting technology adoption (**????**), these quantitative frameworks typically abstract from the role of IP in obtaining external financing, thereby limiting their ability to analyze the implications of policies that expand access to intangible asset-backed collateral.

We then develop a general equilibrium model of heterogeneous firms that jointly choose capital and R&D investment subject to collateral constraints to interpret the empirical findings. In the model, firms invest in R&D to raise future productivity, which in turn leads to the accumulation of pledgeable IP.[1] The framework therefore features endogenous productivity dynamics and endogenous net-worth accumulation. Allowing IP to serve as collateral has two key implications. First, it expands firms' borrowing capacity and mitigates capital misallocation arising from collateral-based borrowing constraints. Second, because IP relaxes future financing constraints, it increases firms' incentives to accumulate pledgeable IP by undertaking more R&D in the present. The first channel is standard in models in which borrowing is collateralized by tangible assets (**??????**), whereas the second channel is specific to financing backed by IP collateral.

The model delivers sharp theoretical predictions that rationalize our empirical findings and discipline our quantitative decomposition. On the capital-allocation side, when the collateral constraint binds, it is tighter for firms with lower net worth—equivalently, firms face a higher shadow cost of external finance and therefore a higher marginal revenue product of capital (MRPK). Moreover, conditional on net worth, the constraint can become tighter as productivity increases if the induced increase in desired capital outpaces the increase in the collateral value of IP.[2] These comparative statics imply that high-productivity but low-net-worth firms

[1]We use "intellectual property" to refer to legally protectable assets (e.g., patents and trademarks) that can be pledged as collateral, and "intellectual capital" to refer to the broader stock of firm knowledge that improves production efficiency. In the model, R&D builds intellectual capital, which generates intellectual property that can be pledged for loans.

[2]Our estimation confirms the assumption that the induced increase in desired capital outpaces the rise in IP collateral value following an increase in productivity.

are systematically more constrained and therefore display higher MRPK, consistent with the empirical patterns we document. On the innovation side, R&D is financed internally and features nonconvex adjustment due to a fixed cost of initiating R&D, together with convex variable costs in the targeted R&D intensity. This structure yields a threshold rule on the extensive margin: for a given level of current productivity, a firm conducts R&D if and only if its net worth exceeds a productivity-dependent cutoff. Conditional on R&D participation, the model implies that R&D intensity increases with net worth. These implications for R&D participation and intensity are consistent with the data and are central moments targeted in estimation. In the aggregate, the model implies that expanding IP-backed financing raises TFP through two channels. First, it generates static gains by improving capital allocation as collateral constraints relax, as in quantitative frameworks with exogenous productivity (e.g., **??????**). Second, it generates dynamic gains as firms invest in R&D and accumulate intangible collateral that supports future borrowing (e.g., **???**).

IP-backed lending differs from a conventional expansion of credit secured by tangible collateral in two key respects. First, on the capital-allocation side, increasing IP pledgeability expands credit to the most constrained firms—those with high productivity but low net worth—by more than a comparable increase in tangible-collateral pledgeability, which disproportionately relaxes constraints for firms that already hold substantial physical capital. Second, on the innovation side, IP-backed financing strengthens incentives to invest in R&D because R&D generates pledgeable IP that relaxes future borrowing constraints—a mechanism absent from policies that expand lending against tangible collateral. As a result, higher IP pledgeability amplifies innovation through a collateral-creation channel: firms invest in R&D both to raise productivity and to accumulate assets that can be pledged in future credit markets.

We estimate the model using the method of simulated moments. We target moments that capture how financial constraints shape capital allocation and both the extensive and intensive margins of R&D, thereby jointly identifying the key structural parameters. A central object is the parameter governing the pledgeability of IP before and after the policy. This parameter determines the extent to which firms can leverage IP to expand borrowing capacity and, in turn, the quantitative impact of the IP-backed financing policy. Our identification strategy exploits the policy-induced shift in IP pledgeability: in the model, higher pledgeability relaxes borrowing constraints and increases the return to R&D by expanding the stock of future pledgeable IP, implying a monotone relationship between R&D participation and the pledgeability parameter. We therefore use the DID estimate of the policy's effect on R&D participation in pilot cities as the key moment that pins down the change in IP pledgeability. Combining this quasi-natural experiment with the model's structure yields a structural estimate of IP pledgeability in China. To our knowledge, estimating an intangible-collateral pledgeability parameter using policy-induced variation is new in the quantitative literature on financial frictions, which typically calibrates collateral parameters and focuses on tangible assets (e.g., **????**).

We validate the estimated model by showing that it is consistent with several key moments not targeted in estimation. First, the model matches the average ratio of R&D expenditures to value added among R&D-performing firms. Second, although the estimation targets only the extensive-margin response of R&D to the policy, the model's implied intensive-margin response to the IP-backed financing policy closely aligns with the corresponding DID evidence. Third, the model reproduces the cross-sectional patterns in the data whereby firms with lower net worth and higher productivity exhibit higher MRPK and therefore face tighter borrowing constraints.

The estimates indicate that firms faced tight financial constraints prior to the IP-backed financing reform. In particular, the implied pledgeability of tangible assets is substantially below the U.S. benchmark. Our counterfactual suggests that a nationwide adoption of IP-backed financing would raise aggregate productivity through two channels: an immediate, static gain from improved capital allocation, and a gradual, dynamic gain as the reform stimulates R&D investment. The transition dynamics imply that the static TFP gains are sizable and front-loaded—9.50% in the short run, rising modestly to 9.66% in the long run—whereas the dynamic gains begin at zero and build to about 4.41%. Accounting for these innovation responses is therefore essential for quantifying the long-run effects of policies that expand the pledgeability of intangible collateral.

Finally, we use the estimated model to conduct a counterfactual comparison between IP-backed financing and a conventional expansion of lending secured by tangible assets. Specifically, we consider a policy that raises the pledgeability of tangible collateral to the U.S. benchmark. The counterfactuals show that expanding IP-backed financing delivers substantially larger productivity gains than this traditional collateral-based reform. The difference reflects two features of IP collateral emphasized by the model. First, it better aligns credit expansion with firms' financing needs by directing relatively more borrowing capacity toward high-productivity firms that are disproportionately constrained under tangible-collateral lending. Second, it strengthens innovation incentives through a collateral-creation channel: firms invest in R&D not only to raise productivity, but also to accumulate pledgeable intangible assets that support future borrowing. Quantitatively, matching the productivity gains from IP-backed financing using tangible-collateral policies alone would require an implausibly large increase in leverage. These findings highlight the potential of financial innovations that mobilize intangible assets, particularly in economies where limited tangible collateral impedes efficient capital allocation and innovation.

Our results contribute to the understanding of the aggregate implications of financial frictions. Existing quantitative macro and corporate finance studies on financial frictions typically focus on understanding the impact of financial frictions on capital misallocation, corporate leverage, and aggregate productivity (e.g., **?????????**)[3]. These studies often abstract from the role of innovation investment decisions. Therefore, financial frictions only affect aggregate productivity through impairing the efficient allocation of capital across firms. In contrast, our model incorporates firms' innovation investment so that financial frictions can affect aggregate productivity by reducing innovation investment.

This paper also contributes to the literature on the effects of financial frictions on innovation and the growing macroeconomic importance of intangible capital. Existing quantitative studies have shown that financial constraints reduce innovation investment, distort resource allocation, and lower aggregate productivity (see, e.g., **?????**). Furthermore, a growing body of work highlights the unique economic properties of intangible capital and its rising share in firm value (e.g., **???**). These quantitative frameworks incorporate endogenous productivity with innovation investment subject to financial constraints. We complement this strand of literature by: (1) incorporating the use of intangible assets as collateral which strengthens firms' incentives to innovate for the purpose of accumulating pledgeable intangible assets, as in **?**; and (2) building a meaningful micro-founded R&D cost function to capture the patterns of firm-level R&D data, as in some dynamic R&D investment models (e.g., **?**). This enables us to accurately analyze the aggregate implications of policies that expand access to IP-backed

[3]**?** allow financial frictions to influence the firms' entry into the modern sector which entails higher productivity.

collateral in the modern economy.

More broadly, our study contributes to a large body of empirical studies documenting that financial market development stimulates R&D investment (e.g., see ????????). In particular, policies that expand access to intangible asset-backed collateral can stimulate R&D investment (??). Focusing on China's listed firms, ? find that the IP-backed financing policy in China significantly increases firms' patenting activities and shifts firms from secrecy-based innovation to patent-based innovation. We contribute to this literature in two important ways. First, our empirical evidence complements their findings by showing that the policy also increases R&D investment and productivity for a more representative sample of Chinese firms. Second, we provide a quantitative framework to analyze the aggregate effects of the IP-backed financing policy. In particular, we incorporate a micro-founded R&D cost function that captures the realistic pattern of both the extensive and intensive margins of R&D investment.

The remainder of this paper proceeds as follows. Section 2 provides institutional background for China's IP-backed financing policy and describes the data. Section 3 presents a DID analysis examining the effects of the IP-backed financing policy on firm-level performance and cross-firm capital allocation efficiency. Section 4 develops a heterogeneous-firm model with endogenous R&D and capital investment decisions. Section 5 details the structural estimation of the model. Section 6 presents quantitative analysis. Section 7 concludes.

2 Data Description

Our work draws on three primary data sources: (1) policy data indicating which cities began implementing IP-backed financing policies in specific years; (2) Annual Surveys of Industrial Production (ASIP) from 2005 to 2007, which provide information on firms before the policy implementation; and (3) Administrative Income Tax Records (AITR) from 2012 to 2014, which capture firm-level information after the policy was enacted. Details for each dataset are provided below.

2.1 Policy Data

Institutional Background. Although the 1995 Law of Guarantee legally permitted the use of intellectual property as collateral, lending against IP remained negligible prior to 2009.[4] Indeed, as shown in Figure 1, the average number of patent-pledged loans per city hovered around zero during this period. Two main factors explain this limited adoption: First, the absence of standardized IP valuation tools and weak enforcement mechanisms, which heightened banks' risk aversion; Second, the generally low quality of patent portfolios among Chinese firms during their early stages of technological development.[5]

Pilot Program Design. In 2009, SIPO introduced a national pilot program designating selected cities as experimental sites for patent pledge financing. SIPO did not provide direct lending but instead focused on designing institutional arrangements and incentives to facilitate the IP-backed financing by small-and-medium sized enterprises (SMEs hereafter).[6] Core interventions included: (1) issuing guidelines to establish local pledge systems with risk compensation and valuation mechanisms, (2) coordinating with the People's Bank of China and local governments to streamline loan processing, (3) training banks, firms, and agencies in patent assessment and loan procedures, and (4) subsidizing banks via interest discounts and risk-sharing pools.[7]

Policy Implementation. Following China's established policy experimentation model, pilot cities were selected through an application process. Local governments submitted proposals, which SIPO evaluated based on three criteria: (i) evidence of basic IP financing infrastructure and patenting capacity, (ii) readiness to implement risk-sharing mechanisms and establish bank

[4] According to SIPO's 2008 Annual Report, fewer than 5% of SMEs relied on IP collateral.

[5] Concerns about patent quality gained traction in the late 1990s, as China prepared for WTO accession. The 2000 Patent Law Amendment expanded scope and strengthened enforcement to align with TRIPS. WTO entry in 2001 exposed Chinese firms to international competition and patenting standards. While multinational corporations rapidly expanded filings in China, domestic firms often emphasized patent quantity over quality, partly driven by subsidies tied to patent counts (**?**).

[6] Because the program mainly targeted SMEs, analyses relying on listed firms alone understate its aggregate impact.

[7] For example, Beijing allocated 50 million RMB to risk-sharing pools in 2010.

partnerships, and (iii) demonstrated willingness of local banks to accept patents as collateral.[8] Given that local economic performance is closely tied to bureaucratic promotion, city governments had strong incentives to secure pilot designation and ensure program success (**??**). To conduct regression analysis, we do not separately identify the effects of these individual measures. Instead, we treat them as a bundle of policies that collectively expanded access to IP-backed financing. Online Appendix OA-B presents a simple asymmetric information model illustrating how such interventions can increase banks' willingness to lend to financially constrained firms.

Pilot approvals occurred in several waves. Fourteen cities were designated in 2009, followed by four in 2010, one in 2011, seven in 2012, and one in 2013 (see Table OA-3 and Figure OA-1 in Online Appendix OA-A.2).[9] In our baseline analysis, we focus on the 2009–2010 pilot cities as treated and exclude the later cohorts (2011–2013) to ensure a clean and credible DID design. The early cohorts cover 18 cities, whereas only 9 cities were designated in 2011–2013, so including the latter adds little precision but risks bias. Later-treated cities would act as "controls" before their treatment, potentially introducing contamination and negative weights in a staggered DID setting (**?**). Moreover, our post-treatment window (2012–2014) captures mature exposure for the 2009–2010 pilots but only early or partial exposure for the 2011–2013 pilots, mechanically attenuating estimated effects. Excluding these later cohorts therefore yields a more consistent comparison between treated and never-treated cities and strengthens the interpretability of our results.

We exploit China's introduction of IP-backed financing as a quasi-experiment. To construct the policy dataset, we manually collected information on pilot city designation between 2008 and 2014 from the State Intellectual Property Office (SIPO).[10]

Effectiveness of the Policy on Lending. According to the Annual Report 2022 of the China National Intellectual Property Administration (CNIPA), the total amount of patent-backed lending increased from 7.1 billion yuan in 2010 to 401.5 billion yuan in 2022. Among all borrowers using patents as collateral, 99.5% are industrial enterprises. As of 2022, at least

[8]SIPO, Notice on Pilot Work for IP Pledge Financing (2009).

[9]Another large wave of 28 pilot cities occurred in 2016, but we exclude it due to the lack of firm-level data after 2014.

[10]SIPO has been renamed China National Intellectual Property Administration (CNIPA).

18,000 firms have benefited from this policy.

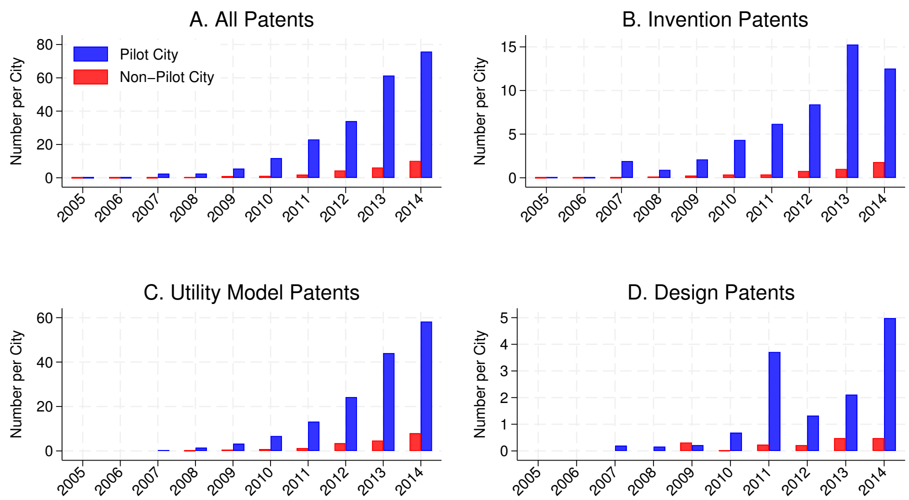


Figure 1: Average Number of Patents per City Over Time: Pilot vs. Non-Pilot Groups

Figure 1 shows that after 2009, the average number of patent-pledged loans increased significantly in pilot cities while remaining at low levels in non-pilot cities, indicating that the policy effectively motivated patent-backed financing.

2.2 Firm Data

Data Sources. Most existing studies rely on publicly listed firms, which provide detailed information on patents and R&D activities (e.g., **?**). However, as highlighted in the policy background, the IP pledge financing program primarily targeted SMEs, which are largely absent from listed-firm datasets. To capture these firms, we require broader coverage of small and medium-sized enterprises. Our focus is on manufacturing firms, consistent with CNIPA's 2022 report that nearly all IP-pledged borrowers were industrial enterprises, most of which operate in manufacturing.[11] We therefore combine two datasets in the baseline analysis: First, we use the Chinese Manufacturing Firms Database (CMFD) for 2005-2007 to capture firm outcomes prior to the policy. Second, we draw on firm-level records from Chinese State Administration of

[11]See the CNIPA *2022 Annual Report*, available at `https://english.cnipa.gov.cn/col/col1336/index.html`. This figure is also reported in the WIPO (2024) publication *Country Perspectives: China's Journey*, available at `https://www.wipo.int/publications/en/details.jsp?id=4709&plang=EN`.

Tax (SAT) for 2011-2014 to examine post-policy firm behavior. These firm-level datasets contain rich information on value added, capital, labor, assets, liabilities, and R&D expenditures. We deflate all nominal variables to 2007 constant prices using the CPI.

Main Variables. We construct four main variables: firm-level R&D expenditure, productivity, net worth, and across-firm capital misallocation. We use R&D expenditure to measure firms' innovation activities. Let $\mathrm{RDX}_{i,t}$ represent the R&D expenditure by firm i in year t, we define a binary indicator $\mathbb{1}\{\mathrm{RDX}_{i,t} > 0\}$ to capture whether firm i in year t undertakes R&D investment, and use $\ln(\mathrm{RDX}_{i,t} + 1)$ to measure the scale of R&D expenditures. Following **?**, we compute a revenue-based measure of firm-level total factor productivity (RTFP) as

$$\mathrm{RTFP}_{i,t} = \frac{\text{Value Added}_{i,t}}{\text{Capital}_{i,t}^{\tilde{\alpha}_k}\text{Labor}_{i,t}^{\tilde{\alpha}_l}},$$

where $\tilde{\alpha}_k$ and $\tilde{\alpha}_l$ are the industry-level capital and labor shares. We further compute firm net worth as total assets minus total liabilities. Finally, we calculate the marginal revenue product of capital (MRPK), defined as the ratio of value added to capital. Capital misallocation is then measured by the dispersion of MRPK, specifically the standard deviation of ln(MRPK) across firms in city c, industry s, and year t, denoted as $\sigma_{c,s,t}^{\text{mrpk}}$. Summary statistics for all variables are reported in Tables OA-1-OA-2 in Online Appendix OA-A.

3 Regression Analysis

Firm-level Outcomes. We begin by analyzing firm-level R&D activities and productivity using the DID approach. The baseline regression specification is as follows:

$$Y_{i,t} = \beta_0 + \beta_1 \cdot \mathrm{Pilot}_i \times \mathrm{After}_{i,t} + \beta_2 \mathrm{Pilot}_i + \beta_3 \mathrm{After}_{i,t} + f_{c(i)} + f_{s(i),t} + v_{i,t}, \tag{1}$$

where $Y_{i,t}$ denotes dependent variables for firm i in year t. We consider three main outcome variables: (1) $\mathbb{1}(\mathrm{RDX}_{i,t} > 0)$ for whether the firm conducts R&D investment, (2) $\ln(\mathrm{RDX}_{i,t}+1)$ for the scale of R&D investment, and (3) $\ln(\mathrm{RTFP}_{i,t})$ for firm's revenue-based productivity. The policy variable Pilot_i equals one if firm i is located in a pilot city implementing the IP-backed financing policy and zero otherwise, and $\mathrm{After}_{i,t}$ equals ones for post-implementation years and zero otherwise. $f_{c(i)}$ is the city-level fixed effects, capturing the city-level time-invariant characteristics, and $f_{s(i),t}$ is the industry-year fixed effects, controlling for industry-specific shocks over time.[12] $v_{i,t}$ is the error term. The coefficient of interest, β_1, captures the causal impact of the policy shock.

Capital Allocation Efficiency. To evaluate how the policy shapes resource allocation, we estimate a DID model at the city-industry level:

$$\sigma_{c,s,t}^{\text{mrpk}} = \delta_0 + \delta_1 \cdot \text{Pilot}_{c,s} \times \text{After}_{c,s,t} + \delta_2 \text{Pilot}_{c,s} + \delta_3 \text{After}_{c,s,t} + f_c + f_{s,t} + v_{c,s,t}, \quad (2)$$

where $\sigma_{c,s,t}^{\text{mrpk}}$ is the standard deviation of $\ln(\text{MRPK}_{i,t})$ across firms in industry s and city c at year t. Lower dispersion indicates more efficient capital allocation. In the baseline, for each city-year, we calculate this statistic at the four-digit industry level.[13] f_c is the city-level fixed effects, and $f_{s,t}$ is the industry-year fixed effects.[14] $v_{c,s,t}$ is the error term. The coefficient δ_1 measures the policy's effect on allocation efficiency.

Baseline Results. Table 1 summarizes the baseline empirical findings. Model 1 shows that the share of firms engaging in R&D increases by 2.45 percentage points in pilot cities relative to non-pilot cities four to six years after implementation. Model 2 considers $\ln(\text{RDX}_{i,t}+1)$ and confirms that the overall scale of R&D expenditures increases, indicating both the extensive and intensive margins of R&D rise significantly. Online Appendix Table OA-9 isolates the intensive margin and shows that, conditional on positive R&D, the policy raises R&D expenditures by roughly 19%. Thus, the program stimulates R&D investment at both margins.

Given that firms conduct more R&D investment, we expect that these firms in pilot cities to experience greater productivity growth. Consistent with this conjecture, Model 3 indicates that RTFP rises by about 28.9% after the policy's implementation. Additional results in Table OA-10 and OA-11 in Online Appendix OA-C.1 show that firms' revenue and labor productivity grow significantly faster in pilot cities, reinforcing the productivity-enhancing effect.

Finally, Model 4 examines capital allocation efficiency. The standard deviation of $\ln(\text{MRPK}_{i,t})$

[12]We experiment with different combinations of fixed effects, the results are stable and reported in Table OA-4 to Table OA-6 in Online Appendix OA-C.1

[13]Results are robust to two-digit industry definitions.

[14]We tried to consider different sets of fixed effects, the results are robust. See Table OA-7 for the results in Appendix OA-C.1.

falls by 0.1, equivalent to a 13% reduction relative to the pre-policy mean.[15] This finding implies that the expanded credit access via patent pledges improves allocative efficiency of capital.

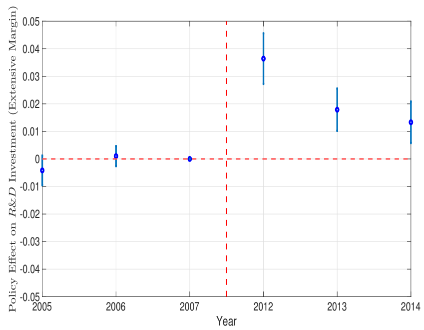
Table 1: Policy Effects on R&D activities, Profitability, and Capital Allocation Efficiency

	Model 1	Model 2	Model 3	Model 4
Dependent Variable	$\mathbb{I}_{\mathrm{RDX}>0}$	$\ln(\mathrm{RDX}+1)$	$\ln(\mathrm{RTFP})$	$\sigma^{\mathrm{mrpk}}_{c,s,t}$
Pilot · After	0.0245***	0.1962***	0.2893***	-0.1000***
	(0.0014)	(0.0106)	(0.0047)	(0.0101)
Observations	1,258,798	1,258,633	753,097	106,830
R-squared	0.080	0.092	0.376	0.184
City FE	Yes	Yes	Yes	Yes
Industry-Year FE	Yes	Yes	Yes	Yes

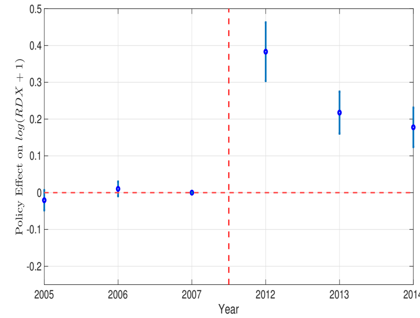
Note: All models include city fixed effects and industry-year fixed effects. Robust standard errors are provided in parentheses. *** p<0.01, ** p<0.05, * p<0.1

Pre-trend Analysis and Placebo Tests. The credibility of our DID estimates hinges on the parallel trends assumption, namely that treated and control cities would have evolved similarly absent the policy. We assess this assumption using an event-study specification that replaces the single "Pilot $\times$ After" indicator with a full set of "Pilot $\times$ Year" interactions. This specification allows us to visualize the dynamic treatment effects over time and, in particular, to examine whether the estimated coefficients for the years prior to treatment are jointly close to zero. As shown in Figure 2, the pre-treatment coefficients are small, statistically insignificant, and do not display systematic trends, providing evidence that treated and control cities followed parallel paths prior to the policy shock.

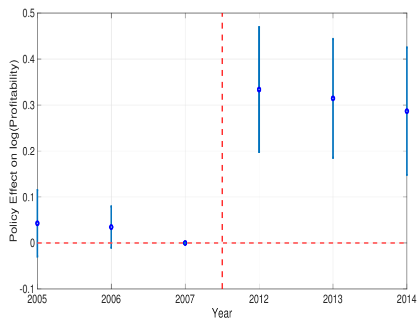
[15]We calculate the 13% by estimating equation (2) with the log of the standard deviation of $\ln(\mathrm{MRPK}_{i,t})$, i.e., $\ln(\sigma^{mrpk}_{c,s,t})$, as the dependent variable. See Online Appendix Table OA-8.



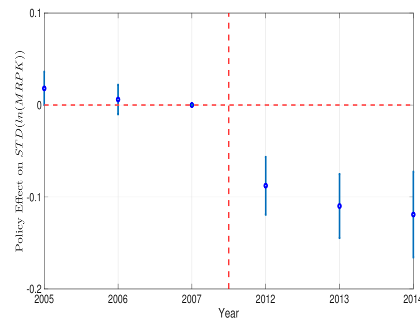
(a) Dependent Variable: $\mathbb{1}\{\text{RDX}_{i,t} > 0\}$



(b) Dependent Variable: $\ln(\text{RDX}_{i,t} + 1)$



(c) Dependent Variable: $\ln(\text{RTFP}_{i,t})$



(d) Dependent Variable: $\sigma^{\text{mrpk}}_{c,s,t}$

Figure 2: Pre-trend Analysis: Event Study Results

Note: The figures plot the coefficients of the interaction terms between the treatment group indicator and year dummies from an event study regression. The omitted category is the year before the policy implementation.

To further probe this assumption, we augment the baseline specification with "City $\times$ Year" interactions, which flexibly control for city-specific shocks or local time trends that might otherwise bias the estimates. As shown in Table 2, the policy effects on R&D and RTFP are more pronounced than in the baseline results, whereas the impact on the dispersion of ln(MRPK) is slightly attenuated. This further reinforces the robustness of our identification strategy.

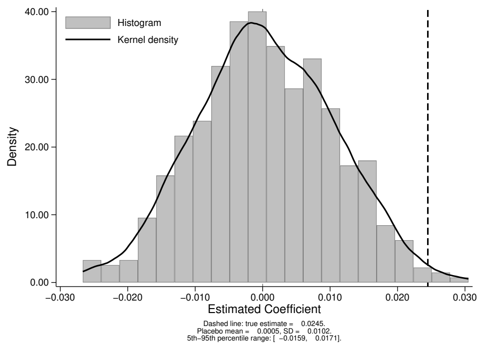
Table 2: The Effects of Policy: Inclusion of City-Specific Time Trends

	Model 1	Model 2	Model 3	Model 4
Dependent Variable	$\mathbb{I}_{\text{RDX}>0}$	$\ln(\text{RDX}+1)$	$\ln(\text{RTFP})$	$\sigma^{\text{mrpk}}_{c,s,t}$
Pilot · After	0.0486***	0.3663***	0.3980***	-0.0627***
	(0.0061)	(0.0447)	(0.0194)	(0.0126)
Observations	1,258,711	1,258,633	753,016	106,705
R-squared	0.086	0.097	0.391	0.441
City Dummy × Year	Yes	Yes	Yes	Yes
City FE	Yes	Yes	Yes	Yes
Industry-Year FE	Yes	Yes	Yes	Yes

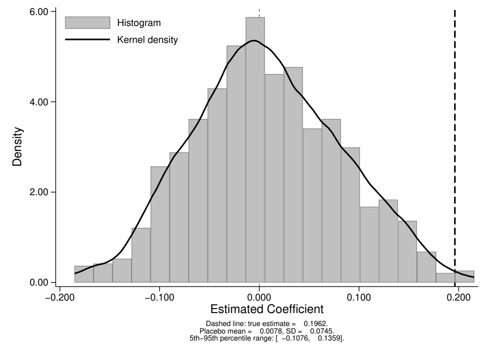
Note: Robust standard errors are provided in parentheses. All models include city fixed effects and industry-year fixed effects. *** p<0.01, ** p<0.05, * p<0.1.

We also conduct placebo tests by randomly assigning treatment status to cities and re-estimating the baseline model 1,000 times. Figure 3 shows that the distribution of placebo estimates centers around zero and is significantly smaller in magnitude than our actual estimates. Online Appendix OA-C.6 provides detailed information on the placebo test.

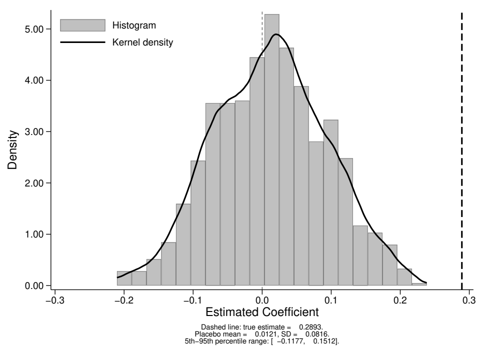
Robustness Checks. To ensure the robustness of our baseline findings, we conduct a series of sensitivity analyses addressing measurement concerns, potential data inconsistencies, and confounding policy shocks. First, we employ alternative proxies for firm performance, including total sales and sales per worker. As reported in Tables OA-10 and OA-11 in Online Appendix OA-C.1, the results remain qualitatively consistent with our primary estimates. Second, we address potential concerns regarding the use of different data sources across the pre- and post-policy periods by utilizing a separate dataset of tax records from 2011 to 2014. In this specification, we restrict the treatment group to cities designated as pilots in 2012 and exclude cities treated in other years to ensure a clean control group. By focusing on a balanced panel of firms, we are able to incorporate firm fixed effects, which accounts for unobserved, time-invariant firm heterogeneity and eliminates concerns regarding inconsistent firm identifiers. The results, presented in Table A.1 in Appendix A, closely mirror our baseline findings. Although the coefficients are slightly smaller in magnitude—a result consistent with the shorter two-year post-treatment horizon compared to the four-to-six-year window in our baseline—the estimated effects remain robust.



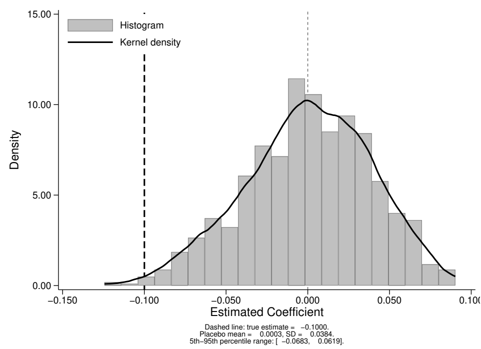
(a) Dependent Variable: $\mathbb{1}\{\mathrm{RDX}_{i,t} > 0\}$



(b) Dependent Variable: $\ln(\mathrm{RDX}_{i,t} + 1)$



(c) Dependent Variable: $\ln(\mathrm{RTFP}_{i,t})$



(d) Dependent Variable: $\sigma^{\mathrm{mrpk}}_{c,s,t}$

Figure 3: Placebo Tests

Note: The figures display the results of the placebo test corresponding to the baseline estimates in Table 1.

Finally, we examine whether our results are confounded by contemporaneous policy initiatives, specifically the National Intellectual Property Demonstration City program.[16] Given that this program's focus on IP infrastructure may overlap with the IP-backed financing pilots, we include a time-varying indicator for demonstration city status in our model. To control for this potential confounder, we collect data on demonstration city designations and incorporate the information on when and whether a city is a demonstration city into our regression models. As shown in Table OA-13 in Online Appendix OA-C.4, our main results are robust to the inclusion of this control, suggesting that the observed effects are driven by the IP-backed financing policy rather than parallel IP-related designations. Online Appendix OA-C.4 provides further

[16]Using Chinese data, **?** show that the effect of intellectual property protection on innovation depends on the level of financial development.

details on this analysis.

Discussion. The regression analysis provides compelling evidence that the IP-backed financing policy stimulates firms' R&D investment and productivity, while also improving the efficiency of capital allocation. Since only a small subset of cities (18 out of 333) were designated as pilots, strong general equilibrium effects at the national level are unlikely. We therefore interpret these reduced-form results as capturing short-run, partial equilibrium effects on firms' R&D decisions, profitability, and allocation efficiency.[17]

While informative, the reduced-form analysis alone cannot address several key questions. What would be the aggregate productivity impact of scaling this policy nationwide, and through which channels would these gains materialize? How would productivity evolve dynamically over time? How do the effects compare with those of traditional collateral policies based on tangible assets? To answer these questions, we develop a structural model in which heterogeneous firms make joint decisions on capital and R&D investment under financial frictions. We estimate the model using the method of simulated moments (SMM). This framework allows us to move beyond partial equilibrium, quantify general equilibrium effects, disentangle productivity improvements from reallocation gains, and conduct counterfactual policy comparisons.

4 Theory

We build a heterogeneous-firm model with endogenous productivity, innovation investment, and financing frictions shaped by tangible and intangible collateral. This model has two important features: First, it incorporates intangible collateral directly into the borrowing constraint. Second, R&D investment contributes to both productivity growth and intangible collateral's accumulation. These features enable us to evaluate how an intangible-collateral-based financing policy shapes firms' innovation choices, productivity trajectories, and the efficiency of capital allocation.

[17] Besides, we also examine city-level wages and find no significant divergence between pilot and non-pilot cities after the policy (see Table OA-14 in Online Appendix OA-C.5). This likely reflects relatively mobile labor across cities.

4.1 Environment

We consider a small open economy populated by a continuum of infinitely-lived entrepreneurs indexed by i.[18] Time is discrete and indexed by $t = 0, 1, 2, \cdots$. Each firm produces a differentiated good using capital and labor, and faces financial frictions that limit borrowing based on the value of its tangible assets and IP. Entrepreneurs can invest in R&D to enhance their productivity, which also increases their intangible collateral value. They make decisions on consumption, labor, capital investment, net worth accumulation, and R&D investment to maximize their expected lifetime utility.

Preferences. Entrepreneur i has time-separable preferences given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_{i,t}^{1-\epsilon}}{1-\epsilon}, \tag{3}$$

where $\beta \in (0, 1)$ is the discount factor, $c_{i,t}$ denotes the consumption goods, and ϵ denotes the coefficient of relative risk aversion. The operator $\mathbb{E}_0$ denotes expectations conditional on information at time 0.

Production and Demand. Each entrepreneur operates a firm that produces differentiated goods using physical capital, labor and intellectual capital. The production function is Cobb-Douglas:

$$q_{i,t} = \phi_{i,t}^{\alpha_\phi} k_{i,t}^{\alpha_k} l_{i,t}^{\alpha_l}, \tag{4}$$

where α_ϕ, α_k and α_l represent the elasticities of output with respect to intellectual capital $\phi_{i,t}$, physical capital $k_{i,t}$, and labor $l_{i,t}$, separately. We impose that $0 < \alpha_\phi < 1$, $0 < \alpha_k < 1$, $0 < \alpha_l < 1$, and $\alpha_k + \alpha_l = 1$ so that the production has constant returns to scale in physical inputs. $q_{i,t}$ represents the firm's output. $\phi_{i,t}$ is the firm's knowledge stock that can be transformed into intellectual property, which evolves stochastically over time and can be influenced by the firm's R&D investment. $\phi_{i,t}^{\alpha_\phi}$ represents the firm's physical productivity (TFPQ as in **?**).

[18] The model can be easily extended to incorporate firm entry and exit, either by assuming an exogenous exit rate or by adding fixed operational costs that induce endogenous entry and exit. However, we cannot observe firm's entry and exit in the sample due to data limitations. Therefore, we choose to not incorporate entry and exit in the current framework and leave it for future study.

Firms own the physical capital and can accumulate it through capital investment with a depreciation of δ. We adopt the convenient assumption that, before firms make decisions for physical capital in period $t+1$, the productivity in period $t+1$ is known at the end (see **???** and **?**, among others).

Firms face constant-elasticity demand:

$$q_{i,t} = p_{i,t}^{-\sigma}, \tag{5}$$

where $\sigma > 1$ is the demand elasticity. We abstract from aggregate shocks to demand, and normalize the demand shifter to be one.[19]

Combining production and demand yields the firm's revenue $\left(\phi_{i,t}^{\alpha_\phi} k_{i,t}^{\alpha_k} l_{i,t}^{\alpha_l}\right)^{\frac{\sigma-1}{\sigma}}$. To distinguish from the underlying physical productivity, we refer to $\phi_{i,t}^{\frac{\sigma-1}{\sigma}\alpha_\phi}$ as the firm's revenue productivity.

R&D Investment and Productivity Dynamics. The logarithm of intellectual capital, $\ln(\phi_{i,t})$, evolves following an AR(1) process:

$$\ln(\phi_{i,t+1}) = \rho \ln(\phi_{i,t}) + \bar{\mu} + \mu_{i,t} + \sigma_\xi \xi_{i,t+1}, \tag{6}$$

where ρ captures the persistence, we restrict that $0 < \rho < 1$, and σ_ξ represents the volatility. The stochastic term $\xi_{i,t+1}$ follows a truncated normal distribution, rescaled to have $\mathbb{E}_t[\xi_{i,t+1}] = 0$ and $\mathrm{Var}(\xi_{i,t+1}) = 1$. Its support is bounded: for some $\bar{\xi} > 0$, $|\xi_{i,t+1}| \leq \bar{\xi}$ almost surely. It captures both idiosyncratic shocks to the quality of existing intellectual capital and the inherent uncertainty of R&D outcomes. Potentially, it could reflect competitive dynamics where new similar patents erode existing patent values or competitor exit enhances surviving firms' patent worth. The drift consists of an exogenous component $\bar{\mu}$ and an endogenous component $\mu_{i,t} \geq 0$, which depends on the firm's R&D effort. We view $\mu_{i,t}$ as the *R&D target* that captures how much productivity growth the firm aims to achieve through R&D activities. The expected productivity conditional on R&D target $\mu_{i,t}$ is:

$$\mathbb{E}_t\left[\phi_{i,t+1} \mid \mu_{i,t}\right] = \exp(\rho \ln(\phi_{i,t}) + \bar{\mu} + \mu_{i,t}) \cdot \mathbb{E}_t\left[\exp(\sigma_\xi \xi_{i,t+1})\right].$$

[19] We show in the Appendix B.1 that the demand function can be derived from the small open economy with a CES demand system. We normalized the aggregate demand shifter to be one.

The expected growth rate of productivity induced by R&D investment is:

$$\frac{\mathbb{E}_t\left[\phi_{i,t+1} \mid \mu_{i,t} > 0\right] - \mathbb{E}_t\left[\phi_{i,t+1} \mid \mu_{i,t} = 0\right]}{\mathbb{E}_t\left[\phi_{i,t+1} \mid \mu_{i,t} = 0\right]} = \exp(\mu_{i,t}) - 1.$$

To reach a chosen R&D target, the firm must incur *R&D expenditures*, denoted as $\chi(\mu_{i,t};\, \phi_{i,t})$. We assume that the R&D cost function has the following form:

$$\chi(\mu_{i,t};\, \phi_{i,t}) = \underbrace{\chi_f(\phi_{i,t})}_{\text{fixed R\&D cost}} + \underbrace{\chi_v(\phi_{i,t},\, \mu_{i,t})}_{\text{variable R\&D cost}}, \tag{7}$$

where $\chi_f(\phi_{i,t})$ is the fixed cost of conducting R&D and $\chi_v(\phi_{i,t},\, \mu_{i,t})$ is the variable cost, both depend on the firm's current productivity. The fixed cost $\chi_f(\phi_{i,t})$ captures the setup costs of R&D activities, such as establishing R&D labs, hiring specialized personnel, and acquiring necessary equipment. We assume that $\chi_f(\phi_{i,t}) > 0$ and $\chi_f'(\phi_{i,t}) \geq 0$, meaning that higher-productivity firms face greater fixed costs, consistent with empirical evidence that high-productivity firms tend to have more complex R&D operations. The variable cost $\chi_v(\phi_{i,t},\, \mu_{i,t})$ captures the ongoing expenses associated with R&D activities, such as salaries for R&D staff, costs of materials and prototypes, and expenses related to testing and development. We assume that $\chi_v(\phi_{i,t},\, 0) = 0$, $\partial\chi_v(\phi_{i,t},\, \mu_{i,t})/\partial\mu > 0$ and $\partial^2\chi_v(\phi_{i,t},\, \mu_{i,t})/\partial\mu^2 \geq 0$, indicating that the variable cost increases with the R&D target at an increasing rate. We also assume that $\partial\chi_v(\phi_{i,t},\, \mu_{i,t})/\partial\phi \geq 0$, meaning that higher-productivity firms face greater variable costs for a given R&D target.

Economically, this productivity-related R&D costs reflect the well-documented asymmetry in innovation opportunities: high-productivity firms, being closer to the technological frontier, face fewer possibilities for imitation and must rely more heavily on original innovation. As a result, sustaining a given productivity growth rate requires disproportionately greater R&D effort relative to firms further from the frontier (**??**). Conversely, low-productivity firms benefit from catch-up opportunities that make convergence less costly. This formulation is consistent with the "two faces of R&D" framework (**??**), which highlights both the innovation-enhancing role of R&D at the frontier and its role in facilitating imitation and absorption for lagging firms. Empirically, it also aligns with evidence that R&D-to-Sales ratio increases with firm size and productivity, implying that larger or more advanced firms must invest disproportionately more to achieve the same rate of productivity growth (e.g., **?**).

Note that the specification (6) abstracts from positive externalities of R&D and the productivity distribution is stationary in the steady state. While R&D spillovers are likely to matter for aggregate productivity, limited data prevent us from identifying them. Consequently, our estimates of policy effects should be interpreted as level effects rather than growth effects in settings where R&D generates spillovers. A richer model could allow $\bar{\mu}$ to depend on the cross-sectional distribution of idiosyncratic productivity $\phi_{i,t}$, generating endogenous growth in the long run. We provide suggestive insights in Section 4.6, where we extend the model to allow for R&D spillovers that generate long-run economic growth, and in Section 6.3, where we quantify the policy effects on the long-run growth rate.

Borrowing Constraints. Entrepreneur i in period t can borrow a one-period bond $b_{i,t}$ at interest rate r_t, which is exogenous under the small open economy assumption. In the macro-finance literature, it is common the amount a firm can borrow is limited by a collateral constraint that depends on its stock of physical capital (see, e.g., **?**, **?** and **?**). However, firms also own intellectual capital—such as patents, trademarks, proprietary technologies, and brand value—that contribute to its production capacity, much like physical capital or labor. In practice, these intangible assets have been widely used as collateral for financing (see, e.g., **???**, among others). To some extent, the intangible assets also represent the firm's potential to generate future cash flows (**?**). For example, patents provide firms with exclusive rights to produce and sell innovations, directly affecting their ability to innovate and scale operations. Similarly, trademarks and brand equity create consumer loyalty and market power, which enhance the firm's profitability and long-term value. Therefore, $\phi_{i,t}$ can be seen as the pledgeable intellectual property, which, much like physical capital, are fundamental to the firm's ability to raise capital and fund expansion.

We extend the traditional physical-capital-based collateral constraint to include intellectual capital, so that the borrowing limit is a function of both the physical capital and intellectual capital:

$$b_{i,t} \leq \tilde{\theta} k_{i,t} + \tilde{\eta} \Psi(\phi_{i,t}), \tag{8}$$

where $\tilde{\theta} \in (0,\, 1)$ and $\tilde{\eta} > 0$ governs the tightness of the collateral constraint for capital stock and intellectual capital, respectively. $\Psi(\phi_{i,t})$ $\tilde{\eta}\Psi(\phi_{i,t})$ is the liquidation value of intellectual capital for creditors.[20] We assume that $\Psi'(\phi_{i,t}) > 0$, meaning that the liquidation value of intellectual capital increases with the level of intellectual capital. The constraint can be derived from an environment in which lenders can seize a fraction $\tilde{\theta}$ of the firm's physical capital and a fraction $\tilde{\eta}$ of the liquidation value of the intellectual capital (see Appendix B.2).

Define the net worth of entrepreneur i at the beginning of period t as $a_{i,t} \equiv k_{i,t} - b_{i,t}$, the collateral constraint can be rewritten as a constraint on capital stock:

$$k_{i,t} \leq \theta a_{i,t} + \eta \Psi(\phi_{i,t}), \tag{9}$$

where $\theta \equiv \frac{1}{1-\tilde{\theta}}$ and $\eta \equiv \frac{\tilde{\eta}}{1-\tilde{\theta}}$. This capital constraint implies that the amount of capital a firm can rent is limited by its net worth and intellectual capital. In the special case where $\eta = 0$ (i.e., $\tilde{\eta} = 0$), the model reduces to the standard collateral constraint based solely on physical assets.[21]

Unlike capital investment, we assume that R&D investment is financed internally. This assumption is consistent with empirical evidence that R&D is primarily funded through internal cash flow rather than external financing (see, e.g., **???**, among others). The entrepreneur's budget constraint in period t is given by:

$$c_{i,t} + a_{i,t+1} + \chi(\mu_{i,t};\, \phi_{i,t}) \leq (\phi_{i,t}^{\alpha_\phi} k_{i,t}^{\alpha_k} l_{i,t}^{\alpha_l})^{\frac{\sigma-1}{\sigma}} - w_t l_{i,t} - (r_t + \delta) k_{i,t} + (1 + r_t) a_{i,t}, \tag{10}$$

where the R&D cost function $\chi(\mu_{i,t};\, \phi_{i,t})$ follows equation (33). The right-hand side of equation (10) captures the entrepreneur's per-period resources in period t. It incorporates the firm's profits from production, which is the revenue minus labor and capital costs, and the return on existing net worth. The left-hand side represents the uses of funds, including consumption, savings (i.e., next period's net worth), and R&D expenditures. Note that as long as the consumption is non-negative, the budget constraint is equivalent to the limited liability condition

[20] The market valuation of intellectual capital, $\tilde{\eta}\Psi(\phi_{i,t})$, reflects its effective pledgeability to creditors. Key determinants include the degree of legal protection and enforceability of intellectual property rights, the technological and market relevance of the innovation portfolio, the extent to which the asset contributes to measurable cash flows, its transferability and existence of secondary markets (e.g., for patents or licenses), and the verification costs lenders face in assessing its value. These factors, together with the quality of financial institutions and the development of intellectual property markets, shape the liquidation value that lenders can recover.

[21] **?** discusses the form of capital constraint $k_{i,t} \leq \theta(\phi_{i,t}) a_{i,t}$, where asset pledgeability depends on productivity. However, in his setting, firms are unable to affect productivity via R&D, whereas our model endogenizes productivity growth through innovation investment.

that the dividend payout is non-negative (see, e.g., **?** and **?**).

In sum, financial frictions are captured by both the borrowing constraint and the internal financing requirement for R&D. The collateral constraint limits the amount of capital a firm can rent based on its net worth and intellectual capital, thereby restricting its ability to scale operations. The internal financing requirement for R&D reflects the empirical observation that innovation activities are often funded through retained earnings rather than external debt or equity. This influences the entrepreneur's allocation of resources among consumption, savings, and innovation investment.

4.2 Dynamic Problem and Equilibrium

Recursive Problem. Given the path of wages and the interest rate, entrepreneur i decides on consumption $c_{i,t}$, savings $a_{i,t}$, R&D investment $\mu_{i,t}$, capital input $k_{i,t}$, and labor input $l_{i,t}$ to maximize lifetime utility. In recursive form, the entrepreneur's problem can be written as:

$$V_t(a,\phi) = \max_{c\geq 0,\, a'\geq a_{\min},\, \mu\geq 0} \left\{ \frac{c^{1-\epsilon}}{1-\epsilon} + \beta \mathbb{E}_t V_{t+1}(a',\phi') \right\} \tag{11}$$

$$\text{s.t.} \quad c + \chi(\mu;\, \phi) + a' = y_t(a,\, \phi)$$

$$\ln(\phi') = \rho \ln(\phi) + \bar{\mu} + \mu + \sigma_\xi \xi', \tag{12}$$

where the per-period resource function $y_t(a,\phi)$ is defined as:

$$y_t(a,\phi) = \pi_t(a,\phi) + (1+r_t)a \tag{13}$$

$$\pi_t(a,\phi) = \max_{k,\, l} \left\{ (\phi^{\alpha_\phi} k^{\alpha_k} l^{\alpha_l})^{\frac{\sigma-1}{\sigma}} - w_t l - (r_t+\delta)k \right\} \tag{14}$$

$$\text{s.t.} \quad k \leq \theta a + \eta \Psi(\phi).$$

To ensure the value function remains well-defined when $\epsilon \geq 1$, we restrict the domain of net worth to exclude zero. Specifically, we impose $a, a' \geq a_{\min}$, where $a_{\min} > 0$ is an arbitrarily small constant. Economically, this assumption implies that firms must retain a minimum level of net worth to remain operational.[22] We also impose $\beta(1+r) \leq 1$.

Competitive Equilibrium. A *competitive equilibrium* consists of value functions $\{V_t(a,\phi)\}_{t=0}^{\infty}$, endogenous wage and exogenous interest rate sequences $\{w_t, r_t\}_{t=0}^{\infty}$, and policy functions for

[22] This assumption is not critical for our quantitative results and is not required when $0 \leq \epsilon < 1$.

consumption $\{c_t(a,\phi)\}_{t=0}^{\infty}$, savings $\{a_{t+1}(a,\phi)\}_{t=0}^{\infty}$, R&D target $\{\mu_t(a,\phi)\}_{t=0}^{\infty}$, capital input $\{k_t(a,\phi)\}_{t=0}^{\infty}$, and labor input $\{l_t(a,\phi)\}_{t=0}^{\infty}$ such that:

- Given the sequences of wages and interest rates, (i) the value functions and policy functions solve the entrepreneur's optimization problem (11) for all t; (ii) the capital choice $k_t(a,\phi)$ and labor choice $l_t(a,\phi)$ solve the profit maximization problem (14) for all t.
- The labor market clears in each period t:

$$L_t = \int l_t(a,\phi)dg_t(a,\phi), \tag{15}$$

where L_t is the exogenous aggregate labor supply and $g_t(a,\phi)$ is the distribution of entrepreneurs over net worth and productivity at time t.

Steady-State Equilibrium. A *steady-state equilibrium* is a competitive equilibrium where the wage $w_t = w$ and the interest rate $r_t = r$ are constant over time, and the distribution of entrepreneurs $g_t(a,\phi) = g(a,\phi)$ is also constant over time.

We establish sufficient, though not necessary, conditions for the steady-state value function $V(a,\phi)$ to be well-defined and unique. If the R&D target (μ) in the productivity process were exogenously given, this would constitute a standard Bellman equation. In our setting, however, the maximum feasible level of μ may become unbounded as the pre-period resources $y(a,\phi)$ increase. Let $\mu_{\max}(a,\phi) := \chi_v^{-1}(y(a,\phi);\phi)$ be an upper bound of feasible μ at state (a,ϕ) given the budget constraint and any positive fixed R&D cost. Assumption 1 below ensures that the asymptotic behavior of this upper bound remains controlled.

Assumption 1. (Restrictions on the R&D cost function for Models with $\epsilon \in [0,1]$)
Let $\upsilon_\phi = \frac{\tilde{\alpha}\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}$, $\tilde{\alpha}_\ell = \frac{\sigma-1}{\sigma}\alpha_\ell$, $\ell \in \{\phi, l, k\}$.

1. For $\epsilon \in [0,1)$, we assume $\limsup_{a\to\infty} \frac{\exp(\upsilon_\phi \mu_{\max}(a,\phi))}{a} = 0$; $\limsup_{\phi\to\infty} \frac{\exp(\mu_{\max}(a,\phi))}{\phi^{1-\rho}} = 0$. Additionally, if $\epsilon = 0$, we require $\beta(1+r) < 1$.
2. For $\epsilon = 1$, we assume $\limsup_{a\to\infty} \frac{\mu_{\max}(a,\phi)}{\ln a} = 0$; $\limsup_{\phi\to\infty} \frac{\mu_{\max}(a,\phi)}{\ln \phi} = 0$.

This assumption dictates that while the maximum feasible level of μ may diverge as pre-period resources $y(a,\phi)$ increase, its asymptotic behavior is bounded relative to a and ϕ.

Intuitively, this is plausible if R&D costs exhibit sufficient convexity with respect to μ. Consequently, the Weighted Contraction Mapping Theorem (**??**) applies, and the value function is guaranteed to be well-defined and unique within the designated functional space. In Online Appendix OA-D.4.1, the proof of Lemma 4 shows details of the proof.

In what follows, we characterize the optimal decision rules of firms in the steady-state equilibrium.

4.3 Optimal Decision Rules

Labor and Capital Choices. Firms solve the profit maximization problem embedded in (14) to determine their optimal choices of capital and labor, given their net worth a and productivity ϕ. The (constrained) optimal capital and labor choices are:[23]

$$l^*(a,\,\phi) = \frac{\alpha_l}{\alpha_k w}\Gamma(w)M(a,\,\phi)^{-\sigma(1-\tilde{\alpha}_l)}\phi^{(\sigma-1)\alpha_\phi} \tag{16}$$

$$k^*(a,\,\phi) = \Gamma(w)M(a,\,\phi)^{-\sigma\tilde{\alpha}_k}\phi^{(\sigma-1)\alpha_\phi}, \tag{17}$$

where $\Gamma(w) = \left(\frac{\tilde{\alpha}_k^{1-\tilde{\alpha}_k}\tilde{\alpha}_l^{\tilde{\alpha}_l}}{w^{\tilde{\alpha}_l}}\right)^\sigma$, $\tilde{\alpha}_\ell = \frac{\sigma-1}{\sigma}\alpha_\ell$, $\ell \in \{\phi, l, k\}$. $M(a,\,\phi)$ is the shadow price of capital or the marginal revenue product of capital (MRPK hereafter) given as:

$$M(a,\phi) \;=\; \max\left\{ r+\delta,\;\; \frac{\Gamma(w)^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}}\phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_l}}}{[\theta a+\eta\Psi(\phi)]^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}}}. \right\}. \tag{18}$$

From equation (18), we see that the shadow price of capital depends on both the firm's net worth a and intellectual capital ϕ. When the borrowing constraint is not binding, the shadow price of capital equals the rental cost of capital, $r+\delta$. However, when the constraint binds, the shadow price exceeds $r+\delta$ and is decreasing in net worth a. This reflects that firms with higher net worth face looser borrowing constraints. However, for the intellectual capital ϕ, it affects the shadow price of capital in two opposing ways. On one hand, a higher ϕ increases the firm's productivity and revenue, thereby increasing the marginal revenue product of capital. On the other hand, a higher ϕ also relaxes the borrowing constraint by increasing the value of intangible collateral, which tends to reduce the shadow price of capital. The net effect of ϕ on the shadow price of capital is therefore ambiguous and depends on parameter values.

[23] See the derivation details of the optimal choices of capital and labor and the MRPK in Appendix C.1 (or Online Appendix OA-D.2).

Let $k^u(\phi)$ denote the capital demand when the firm is unconstrained, which corresponds to the shadow price $M = r + \delta$. From equation (17), we have

$$k^u(\phi) = \Gamma(w)(r+\delta)^{-\sigma}\phi^{(\sigma-1)\alpha_\phi}, \tag{19}$$

which implies the elasticity $\frac{d\log k^u(\phi)}{d\log(\phi)} = \frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}$.

The following condition guarantees that the impact on fund demand outweighs the impact of relaxing collateral-related financial constraints.

Assumption 2. The parameters of the production technology and the intangible collateral function $\Psi(\phi)$ satisfy:

$$\frac{d\log k^u(\phi)}{d\log(\phi)} > \sup_\phi \frac{d\log\Psi(\phi)}{d\log\phi},$$

Define $\hat{a}(\phi)$ as the threshold level of net worth below which the borrowing constraint becomes binding. The following proposition characterizes the comparative statics of the borrowing constraint with respect to net worth and productivity.

Proposition 1 (Properties of the Borrowing Constraint)**.** (i) Under Assumption 2, the threshold $\hat{a}(\phi)$ is strictly increasing in ϕ; (ii) In the binding region, i.e., when $a < \hat{a}(\phi)$, the MRPK $M(a,\phi)$ is strictly decreasing in a, and, under Assumption 2, strictly increasing in ϕ.

Proof. See the proof in Appendix C.2 (or more details in Online Appendix OA-D.3). □

The intellectual capital affects borrowing tightness through two opposing channels: a productivity channel and a collateral channel. To see it clearly, define the elasticity of $\hat{a}(\phi)$ with respect to ϕ as:

$$\frac{d\log\hat{a}(\phi)}{d\log\phi} = \frac{k^u(\phi)}{\theta\hat{a}(\phi)}\left(\frac{d\log k^u(\phi)}{d\log\phi} - \frac{\eta\Psi(\phi)}{k^u(\phi)}\frac{d\log\Psi(\phi)}{d\log\phi}\right). \tag{20}$$

In (20), the first term inside the bracket, $\frac{d\log k^u(\phi)}{d\log(\phi)}$, captures how intellectual capital shifts the firm's unconstrained demand for physical capital. Increasing intellectual capital enhances the marginal product of physical capital, which increases the demand for capital and tightens the borrowing constraint. Higher ϕ raises the marginal product of physical capital, increasing desired investment and, for a given net worth a, requiring more external finance. This effect raises the cutoff $\hat{a}(\phi)$ and therefore expands the region in which the borrowing constraint binds (i.e., $a \leq \hat{a}(\phi)$).

The second term captures the collateral channel through which intellectual capital relaxes borrowing constraints. The ratio $\frac{\eta\Psi(\phi)}{k^u(\phi)}$ scales the quantitative importance of this channel: it is larger when the collateralizable component of intellectual assets, $\eta\Psi(\phi)$, is relatively high. Since $\frac{d\log\Psi(\phi)}{d\log\phi} > 0$, an increase in ϕ raises the collateral value of intangible assets, which relaxes the constraint and tends to lower $\hat{a}(\phi)$. Equivalently, higher ϕ shrinks the constrained region $a \leq \hat{a}(\phi)$ and reduces the likelihood that the borrowing constraint binds.

The net effect depends on the relative strength of these two forces. Under Assumption 2, the productivity channel dominates, implying $\frac{d\log\hat{a}(\phi)}{d\log\phi} > 0$. Moreover, the impact of ϕ on $\hat{a}(\phi)$ is attenuated when θ is larger, since a higher θ reduces the scaling term $\frac{k^u(\phi)}{\theta\hat{a}(\phi)}$. By contrast, a higher η strengthens the collateral channel by increasing the weight on the intangible collateral $\Psi(\phi)$, pushing the elasticity of $\hat{a}(\phi)$ with respect to ϕ downward. This proposition clarifies the policy mechanism behind IP-backed financing reforms. This policy strengthens the collateral channel by raising creditors' ability to seize and redeploy intangible assets or increasing the liquidation value.

This monotonicity result for MRPK—decreasing in net worth and increasing in productivity—has clear empirical counterparts. In Section 5.6, we document this pattern in the data (see Table 6).

R&D and Net Worth Choices. Due to the fixed cost associated with R&D, the firm's optimal R&D investment follows a discrete-continuous choice structure. In each period, the firm first decides whether to incur the fixed cost to participate in R&D. Conditional on participation, it then determines the optimal investment intensity. We characterize this problem using the upper envelope of two value functions: $V^0(a,\phi)$, which represents the value of abstaining from R&D, and $V^1(a,\phi)$, which represents the value of paying the fixed cost and choosing R&D expenditure optimally.

The entrepreneur compares these two possibilities and chooses the one that implies higher value:

$$V(a,\,\phi) \equiv \max\{V^0(a,\,\phi),\,V^1(a,\,\phi)\} \tag{21}$$

The entrepreneur chooses to invest in R&D, i.e., $\mu > 0$, if $V^1(a,\,\phi) > V^0(a,\,\phi)$. If the firm chooses to invest in R&D, the optimal R&D investment $\mu(a,\phi)$ and net worth $a'(a,\phi)$ can be derived from the first-order conditions of the value function $V^1(a,\, \phi)$, which are given by:

$$
\begin{aligned}
U'[y(a,\phi) - a' - \chi(\mu;\, \phi)] \cdot \chi'(\mu;\, \phi) =& \beta \int_0^\infty V(a',\, \phi') \frac{\partial p(\phi'|\phi,\, \mu)}{\partial \mu} d\phi' \\
=& \beta \int_0^\infty \frac{\partial V(a',\, \phi')}{\partial \log(\phi')} p(\phi'|\phi,\, \mu) d\phi' \qquad (22) \\
U'[y(a,\phi) - a' - \chi(\mu;\, \phi)] =& \beta \int_0^\infty \frac{\partial V(a',\, \phi')}{\partial a'} p(\phi'|\phi,\, \mu) d\phi', \qquad (23)
\end{aligned}
$$

where $p(\phi'|\phi, \mu)$ denotes the conditional density function of next-period productivity, which follows the stochastic process specified in equation (6). Equation (22) equates the marginal utility cost of R&D spending to its expected marginal benefit: a higher μ shifts the distribution of future productivity and hence raises expected continuation value. The second equality in equation (22) further demonstrates that the marginal benefit of R&D investment is proportional to the expected marginal value of future productivity, weighted by the density of next-period productivity. Equation (23) is the Euler equation for net worth: the left-hand side reflects the marginal cost of saving (i.e., next period's net worth), and the right-hand side represents the expected marginal value of additional net worth in the next period.

Under Assumption 1 and Assumption 2, the following proposition characterizes the optimal R&D investment decisions. Let $\underline{\phi}$ denote a lower bound of ϕ.[24]

Proposition 2 (Optimal R&D Investment Decisions)**.** Fix $\phi \geq \underline{\phi}$.

(i) Extensive margin. There exists a cutoff $\underline{a}(\phi)$ such that the firm undertakes R&D if and only if $a \geq \underline{a}(\phi)$.

(ii) Intensive margin. Conditional on undertaking R&D, the optimal R&D intensity $(\mu^*(a, \phi))$ is weakly increasing in net worth a. If $\epsilon > 0$, then the optimal R&D intensity $(\mu^*(a, \phi))$ is *strictly* increasing in a.

Proof. The proof of this proposition relies on two key lemmas. First, we establish the monotonicity and supermodularity of the per-period resource function $y(a, \phi)$ with respect to (a, ϕ), which ensures that the marginal benefit of R&D investment is increasing in a. Second, we show that consumption increases in a conditional on the discrete R&D choice, implying that the utility cost of investment is decreasing in a. See Appendix C.3 for a sketch of the proof (or Online Appendix OA-D.4 for the complete derivation). □

[24]Under the conditions $0 < \rho < 1$, $|\xi'| \leq \bar{\xi}$, and $\mu \geq 0$, the existence of such a bound is guaranteed.

The proposition formalizes the central role of net worth in the firm's R&D investment decisions. Conditional on current productivity, net worth affects the marginal benefit of R&D through two channels. First, given concave preferences (implying diminishing marginal utility of consumption), the shadow value of internal funds is decreasing in a (weakly so if risk neutral). As net worth rises, the utility cost of diverting an additional unit of resources toward R&D rather than consumption—represented by the left-hand side of (22)—declines (weakly so if risk neutral), making R&D investment relatively more attractive. Second, a higher level of a implies a lower probability of facing binding constraints in the future, thereby facilitating a higher level of future capital stock. This higher capital stock increases the marginal return to innovation ϕ, raising the marginal benefit of R&D. Both effects reinforce the incentive to increase R&D as a rises.

In the presence of fixed R&D participation costs, the firm chooses not to invest in R&D when net worth is low, either because the investment is financially infeasible or because the associated utility cost is prohibitively high. As net worth increases, the firm eventually reaches a threshold $\underline{a}(\phi)$ where the benefit of R&D exceeds the cost, prompting entry into R&D activities. Due to the non-convexity of these costs, the optimal R&D investment exhibits a discontinuity (a jump) at this threshold. Figure 4 illustrates this optimal R&D investment policy (the black curve) with respect to net worth for two different productivity levels.[25] Beyond the threshold, higher net worth further incentivizes R&D investment. Overall, this proposition is consistent with the empirical finding that financially constrained firms are less likely to engage in R&D (**??**) and aligns with the data moments reported in Table 5.

The relationship between current net worth and savings (next period's net worth) is primarily driven by consumption smoothing motives. When current productivity is low—and thus cash flow is limited—firms tend to dissave, as indicated by $\ln(a'/a) < 0$ (the red curve in the left panel of Figure 4). This occurs because firms anticipate that productivity will increase in the future.[26] Conversely, when current productivity is high (right panel), firms with low net worth accumulate assets to smooth future consumption, resulting in $\ln(a'/a) > 0$. As net worth rises, the rate of asset accumulation declines, eventually turning negative to align with

[25]Figure OA-3a in Online Appendix OA-F.2 illustrates the optimal R&D investment policy with respect to net worth for additional productivity levels.

[26]Note that productivity is mean-reverting even in the absence of R&D.

the interest rate. A distinct feature of this model is the discontinuity in the savings policy: when firms begin to invest in R&D, next period's net worth drops sharply due to the non-convex costs and the discrete jump in R&D expenditure. In this context, R&D investment effectively serves as an alternative form of savings.

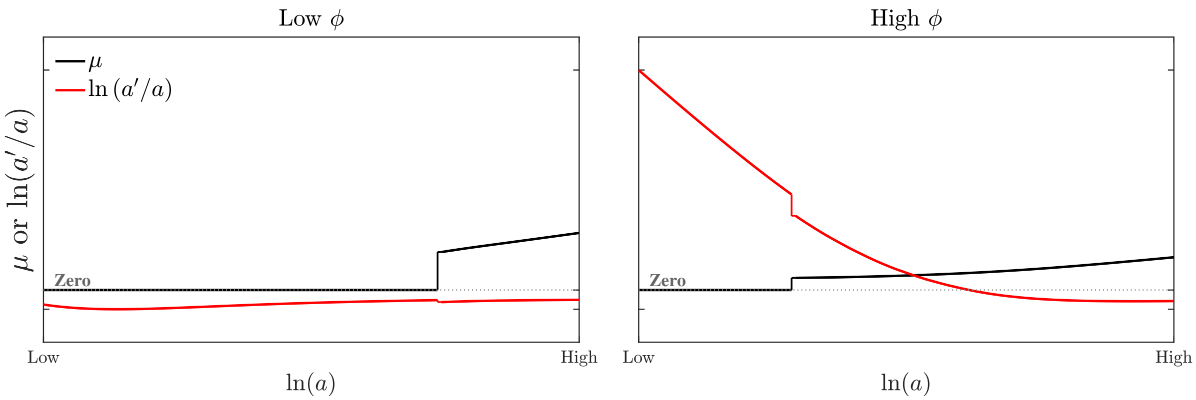


Figure 4: Optimal R&D and saving with respect to net worth: low vs. high Productivity

Note: the scales of y-axis for μ and $\ln(a'/a)$ are the same.

Conditional on the firm's net worth, the effect of current productivity on the optimal R&D choice is more nuanced. On the one hand, the left-hand side of equation (22) implies that a higher productivity level ϕ raises the firm's contemporaneous resources $y(a, \phi)$. This relaxes the within-period resource constraint and lowers the marginal utility cost of allocating resources to R&D, thereby tending to increase R&D investment. On the other hand, because R&D costs depend on productivity, a more productive firm must incur greater expenditure to attain a given R&D target $\mu(a, \phi)$ (i.e., a given rate of productivity growth). This cost channel works in the opposite direction. As a result, the net effect of ϕ on R&D investment is, in general, ambiguous.[27] This ambiguity is reflected in the model's policy functions. As illustrated in Figure OA-3b in Online Appendix OA-F.2, when net worth is low, optimal R&D target, μ, is

[27]Current productivity ϕ can also affect the marginal benefit of R&D (the right-hand side of equation (22)) through its role in the transition density $p(\phi' \mid \phi, \mu)$. In the special case $\rho = 0$, next-period productivity is independent of current productivity, so ϕ does not directly shift the marginal benefit of R&D through the transition. In that case, only the resource and cost channels described above operate, although the overall effect remains ambiguous. When $\rho > 0$, higher current productivity increases expected future productivity and thus raises the expected marginal value of ϕ', altering the marginal benefit of R&D. The overall response of R&D to ϕ therefore depends on the interaction of these three channels.

weakly increasing in ϕ, whereas for sufficiently high net worth it can become decreasing in ϕ.

4.4 The Effects of η and θ on R&D Investment

We now study how an increase in collateral value—that is, a rise in θ or η—affects R&D investment. Collateral value shapes the optimal R&D choice through two opposing forces. On the one hand, higher collateral value expands borrowing capacity and relaxes the capital constraint. As with an increase in net worth, this relaxation lowers the shadow value of internal funds and therefore reduces the utility cost of diverting resources toward R&D (weakly so under risk neutrality). Moreover, by reducing the likelihood that financing constraints bind in future states, greater collateral supports a higher future capital stock; the implied increase in scale raises the marginal payoff to productivity improvements, thereby increasing the marginal benefit of innovation and strengthening incentives to invest in R&D. On the other hand, higher collateral value increases expected resources over time, potentially shifting the intertemporal allocation of spending. When the induced gain is tilted toward the future,[28] the desire to smooth consumption intertemporally raises desired current consumption and can crowd out contemporaneous R&D expenditures. This consumption-smoothing force is stronger for more risk-averse entrepreneurs, since higher risk aversion increases the value of intertemporal smoothing. In the risk-neutral benchmark, by contrast, consumption-smoothing considerations are absent, so increases in collateral value operate primarily through constraint relaxation and tend to raise R&D investment.

We proceed in two steps. We first present a proposition that characterizes the effect of collateral value on R&D investment in the risk-neutral case. We then use a sequence of numerical experiments to examine how the R&D investment policy function responds to an increase in collateral value across different levels of risk aversion.

Risk-Neutral Entrepreneurs. Under Assumption 1 and Assumption 2, the following proposition characterizes the effect of θ and η on R&D investment decisions.

Proposition 3. (Comparative Statics) Fix $\epsilon = 0$.

[28] For example, even if the capital constraint is slack today, it may bind in the future following adverse productivity shocks; higher collateral then disproportionately raises expected future (rather than current) resources.

1. *Impact of θ on R&D.* Let $1 \leq \theta_L < \theta_H$. For any state $(a, \phi) \in [a_{\min}, \infty) \times [\underline{\phi}, \infty)$, the following properties hold:

   (i) **Extensive margin.** The threshold for undertaking R&D investment satisfies

   $$\underline{a}(\phi; \theta_H) \leq \underline{a}(\phi; \theta_L).$$

   (ii) **Intensive margin.** The optimal R&D target for a firm undertaking R&D satisfies

   $$\mu^*(a, \phi; \theta_H) \geq \mu^*(a, \phi; \theta_L).$$

2. *Impact of η on R&D.* Let $0 = \eta_L < \eta_H$. For any state $(a, \phi) \in [a_{\min}, \infty) \times [\underline{\phi}, \infty)$, the following properties hold:

   (i) **Extensive margin.** The threshold such that a firm undertakes R&D satisfies

   $$\underline{a}(\phi; \eta_H) \leq \underline{a}(\phi; \eta_L).$$

   (ii) **Intensive margin.** The optimal R&D target for a firm undertaking R&D satisfies

   $$\mu^*(a, \phi; \eta_H) \geq \mu^*(a, \phi; \eta_L).$$

Proofs of Proposition 3. We show that the per-period resource function $y(\phi; \cdot)$ is monotone and supermodular in (ϕ, θ) and in (ϕ, η). Setting $\epsilon = 0$ shuts down the countervailing consumption-smoothing force present when $\epsilon > 0$. As a result, variation in collateral value operates primarily through the marginal-benefit channel: higher θ or η increases the marginal benefit of R&D investment. See Appendix C.4 for a sketch of the proof, or Online Appendix OA-D.5 for the complete derivation. □

The risk-neutral condition ensures that firms have sufficient incentives to invest in R&D. In the literature, **??** and **?** also consider the case of risk-neutral firms in analyzing the impact of financial constraints on innovation.

Risk-Averse Entrepreneurs. When entrepreneurs are risk averse (i.e., $\epsilon > 0$), Figures A.1 and A.2 in Appendix D report numerical policy functions for optimal R&D and net worth across different degrees of risk aversion.[29] The results are consistent with Proposition 3 for a broad range of ϵ. As ϵ increases, however, the positive effect of higher θ or η on R&D investment becomes weaker, indicating that the consumption-smoothing channel becomes increasingly important.

Comparing the Effects of θ and η. Both Proposition 3 for risk-neutral case and our numerical experiments for risk-averse cases imply that the two reforms increase R&D investment. Their quantitative effects, however, are highly heterogeneous across firms with different levels of net worth and productivity. The key distinction is that the traditional tangible-collateral parameter θ expands borrowing capacity by increasing the pledgeable value of net worth a in the capital constraint, whereas the IP-backed financing parameter η expands borrowing capacity by increasing the pledgeable value of intellectual capital ϕ. Figure 5 illustrates these differences using numerical examples. An increase in η has the largest effect on firms with high productivity and relatively low net worth, while an increase in θ disproportionately affects firms with high net worth and relatively low productivity.[30]

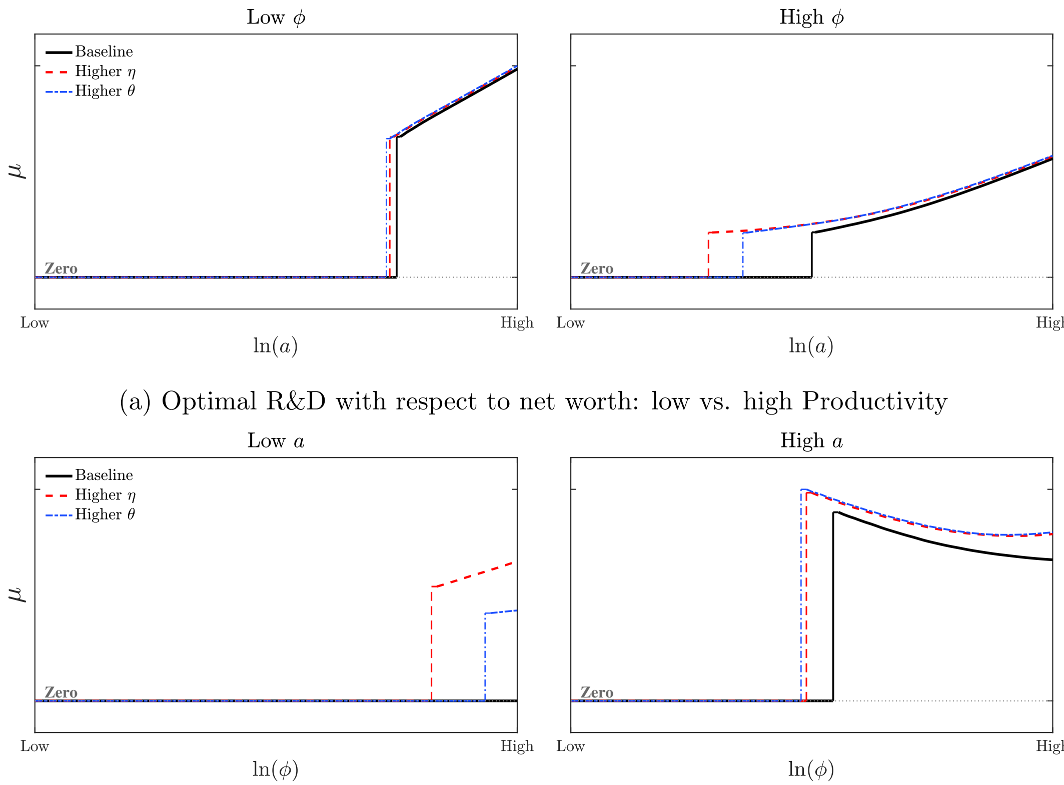
These heterogeneous responses reflect two features of the IP-backed reform. First, a higher η shifts borrowing capacity toward firms with greater intellectual capital, thereby relaxing constraints precisely for firms that are "rich" in ϕ but "poor" in a. Second, by making ϕ more pledgeable, an increase in η strengthens the dynamic incentive to accumulate intellectual capital: firms internalize that higher future ϕ relaxes future borrowing constraints, which raises the marginal benefit of R&D today. This forward-looking collateralization effect is most pronounced for firms with high productivity and low net worth, for which additional pledgeable intellectual capital is particularly valuable. By contrast, the tangible-collateral reform operates primarily through the pledgeability of net worth and therefore benefits firms with higher a more strongly, without directly increasing the collateral value of ϕ.

The aggregate implications of the two reforms depend on the joint distribution of firms' net worth and productivity. In the quantitative analysis, we numerically compare tangible-backed lending and IP-backed financing in terms of their effects on aggregate R&D investment and

[29] In these experiments, we vary ϵ and hold all remaining parameters fixed at their estimated values from Section 5.

[30] Figure OA-5 in Online Appendix OA-F.2 reports additional experiments on a wider grid of (a, ϕ); the qualitative patterns are robust.

productivity.



(a) Optimal R&D with respect to net worth: low vs. high Productivity

(b) Optimal R&D with respect to productivity: low vs. high net worth

Figure 5: Impact of θ and η on Optimal R&D Target

4.5 Aggregate Productivity Effects from Relaxing Financial Constraints

We now turn from firm-level responses to the aggregate implications of the policy. Because aggregate productivity depends on the endogenous evolution of the cross-sectional distribution of firms, we begin by specifying the law of motion of firms distribution.

Law of Motion of State Variables. The law of motion for the distribution of state variables $(a,\, \phi)$ is as follows:

$$G_{t+1}(a', \phi') = \underbrace{\int_{a \leq \underline{a}(\phi),\, \phi} \mathbb{I}\left\{a'(a, \phi) \leq a'\right\} \cdot \Phi_{\xi}(\Delta(\phi, \phi'))dG_t(a, \phi)}_{\text{no R\&D investment}} \tag{24}$$

$$+\underbrace{\int_{a>\underline{a}(\phi),\,\phi} \mathbb{I}\left\{a'(a,\phi)\leq a'\right\}\cdot\Phi_{\xi}(\Delta(\phi,\phi')-\mu(a,\phi))dG_t(a,\phi)}_{\text{with R\&D investment}}$$

where $a'(a,\phi)$ and $\mu(a,\phi)$ are the policy functions for net worth and R&D, respectively, $\Delta(\phi,\,\phi')=\ln(\phi')-\rho\ln(\phi)-\bar{\mu}$. $\Phi_{\xi}(\cdot)$ denotes the CDF of $\sigma_{\xi}\cdot\xi\sim\mathcal{N}(0,\,\sigma_{\xi}^{2})$.

Aggregation and Decomposition of Productivity Effects. The total output is a CES aggregator of all firms' output:

$$Q_t=\left(\int_i q_{i,t}^{\frac{\sigma-1}{\sigma}}di\right)^{\frac{\sigma}{\sigma-1}}, \tag{25}$$

where $q_{i,t}$ is the output of firm i at year t. Let the measure of firms be N. Then the actual total factor productivity (ATFP hereafter) at year t is defined as

$$\text{ATFP}_t=\frac{Q_t}{K_t^{\alpha_k}L_t^{\alpha_l}}$$

where $K_t=N\cdot\int k^*(a,\,\phi)dG_t(a,\,\phi)$ is total capital, and $L_t=N\cdot\int l^*(a,\,\phi)dG_t(a,\,\phi)$ is total labor at year t. Using the (constrained) optimal capital demand (16) and labor demand (17), we can rewrite the actual TFP as:[31]

$$\text{ATFP}_t=N^{\frac{1}{\sigma-1}}\frac{\left[\int\phi^{\sigma\tilde{\alpha}_{\phi}}M(a,\,\phi)^{-\sigma\tilde{\alpha}_k}dG_t(a,\phi)\right]^{\frac{\sigma}{\sigma-1}-\alpha_l}}{\left[\int\phi^{\sigma\tilde{\alpha}_{\phi}}M(a,\phi)^{-\sigma(1-\tilde{\alpha}_l)}dG_t(a,\phi)\right]^{\alpha_k}} \tag{26}$$

In the absence of capital constraints, i.e., $M(a,\phi)=r+\delta$, we obtain efficient TFP (ETFP hereafter):

$$\text{ETFP}_t=N^{\frac{1}{\sigma-1}}\left[\mathbb{E}_t\big(\phi^{\alpha_{\phi}(\sigma-1)}\big)\right]^{\frac{1}{\sigma-1}} \tag{27}$$

where $\mathbb{E}_t\big(\phi^{\alpha_{\phi}(\sigma-1)}\big)=\int\phi^{\alpha_{\phi}(\sigma-1)}dG_t(a,\phi)$ is the $(\sigma-1)\alpha_{\phi}$-th moment of the productivity distribution at year t. The ETFP measures the aggregate productivity when the capital is allocated efficiently across firms given the fundamental productivity distribution. However, due to financial frictions, the actual allocation of capital deviates from the efficient one, leading to a gap between ATFP and ETFP, which we refer to as the static TFP loss from capital misallocation. Similar to **?**, if we assume that ϕ and $M(a,\,\phi)$ follow joint log-normal distribution, ATFP_t can be expressed as

$$\text{ATFP}_t=\exp\left(-\frac{\alpha_k(1-\sigma\tilde{\alpha}_k)}{2}\sigma_M^2\right)\cdot\text{ETFP}_t, \tag{28}$$

[31] See Appendix E for the derivation.

where σ_M^2 is the variance of $\ln(M(a,\, \phi))$. It is clear that the TFP loss from capital misallocation is increasing with the dispersion of $\ln(M(a,\, \phi))$. When the borrowing constraint relaxes, either through increasing θ or η, the dispersion of $\ln(M(a,\, \phi))$ decreases, leading to a higher ATFP.

To see impact of relaxing the collateral constraint on productivity evolution through R&D investment, we can write the actual aggregate TFP as

$$\mathrm{ATFP}_t = \frac{\mathrm{ATFP}_t}{\mathrm{ETFP}_t} \times \mathrm{ETFP}_t$$

The difference in logged ATFP at period t and that at the initial period 0 can be expressed as

$$\begin{aligned} \log(\mathrm{ATFP}_t) - \log(\mathrm{ATFP}_0) = & \underbrace{\log\left(\frac{\mathrm{ATFP}_t}{\mathrm{ETFP}_t}\right) - \log\left(\frac{\mathrm{ATFP}_0}{\mathrm{ETFP}_0}\right)}_{\text{static TFP gains}} \\ & + \underbrace{\log\left(\mathrm{ETFP}_t\right) - \log\left(\mathrm{ETFP}_0\right)}_{\text{dynamic TFP gains}}. \end{aligned} \tag{29}$$

On the right-hand side, the *static TFP gains* arise from improved capital allocation efficiency, and the *dynamic TFP gains* capture the impact of R&D investment on improving ETFP through enhancing firm-level productivity. According to (27), ETFP is aggregated from firm-level productivity distribution. Proposition 3 establishes for the risk-neutral cases, and numerical simulations confirm for the risk-averse case, that promoting tangible-collateral-backed or IP-backed financing stimulates R&D investment. This in turn shifts the distribution of firm's productivity to the right. Therefore, both policies lead to higher dynamic TFP gains.

4.6 Extension: R&D Spillovers and Growth Effects

In the baseline model, firm-level productivity exhibits no spillover effects and there is no long-run economic growth; policy interventions only affect productivity levels. As R&D may also generate growth effects in the presence of positive externalities, we consider extending the model to incorporate these spillovers and long-run economic growth. To do so, we augment the productivity shock process by incorporating a cross-sectional average logged productivity $\overline{\ln \phi_t}$:

$$\ln(\widetilde{\phi}_{i,t+1}) = \rho \ln(\widetilde{\phi}_{i,t}) + \tilde{\mu}_{i,t} + (1-\rho)\overline{\ln \phi}_t + \bar{\mu} + \sigma_\epsilon \epsilon_{i,t+1}, \tag{30}$$

where $\overline{\ln\phi}_t = \mathbb{E}[\ln(\widetilde{\phi}_{i,t})]$, and $\widetilde{\phi}_{i,t}$ and $\tilde{\mu}_{i,t}$ denote firm i's productivity and R&D investment choice, respectively.[32]

Under the spillover process (30), the policy-induced changes in the economy-wide R&D effort $\mathbb{E}[\tilde{\mu}_{i,t}]$ map directly into changes in the long-run growth rate along the BGP.

$$\overline{\ln\phi}_t - \overline{\ln\phi}_{t-1} = \mathbb{E}[\tilde{\mu}_{i,t-1}] + \bar{\mu}, \tag{31}$$

and the aggregate productivity growth rate is

$$\begin{aligned} \log(\text{ATFP}_t) - \log(\text{ATFP}_{t-1}) =& \log(\text{ETFP}_t) - \log(\text{ETFP}_{t-1}) \\ =& \frac{\alpha_\phi \tilde{\alpha}_l}{1-\tilde{\alpha}_k} \left(\mathbb{E}[\tilde{\mu}_{i,t-1}] + \bar{\mu}\right). \end{aligned} \tag{32}$$

This implies that the IP-backed financing policy increases the aggregate TFP growth rate by stimulating firms' R&D investment.

In Online Appendix OA-G.2, we lay out the full growth extension and characterize a balanced growth path (BGP). The key observation is that, after removing the common growth component, the firms' problem in the growth economy can be written in stationary form. In particular, Proposition 8 in Online Appendix OA-G.2.2 establishes an isomorphism between the detrended growth model and the pre-policy stationary baseline model. Since the detrended policy rules coincide with the baseline, we can adopt the baseline solution methods. Furthermore, by leveraging (31), we can evaluate the effects of policy on long-run growth in addition to level effects.

5 Estimation

This section details the estimation procedure of the model [33], which follows three main steps. First, we parametrize the function for the liquidation value of intellectual capital and R&D cost functions. Second, we calibrate the parameters for the demand function, production function, capital depreciation, and interest rate, using information obtained from data or the macroeconomics literature. Third, we then estimate the remaining key parameters related to

[32] Since the data do not allow us to identify the functional form of externalities, we employ this parsimonious specification.

[33] We solve the Bellman equation numerically using the methods detailed in Online Appendix OA-F.1.1.

the R&D cost function, borrowing constraints, productivity processes, and the discount factor using the simulated method of moments (SMM) by minimizing the distance between data moments and model-generated moments.

5.1 Parameterization

We assume the R&D cost function takes the following form:[34]

$$\chi(\mu_{i,t};\, \phi_{i,t}) = \underbrace{\mathbb{1}\{\mu_{i,t} > 0\} \cdot \phi_{i,t}^{\zeta_1} \cdot f}_{\text{fixed cost}} + \underbrace{\gamma \cdot \phi_{i,t}^{\zeta_2} \cdot (\exp(\mu_{i,t}) - 1)}_{\text{linear variable cost}} + \underbrace{\frac{\nu}{2} \cdot \phi_{i,t}^{\zeta_2} \cdot (\exp(\mu_{i,t}) - 1)^2}_{\text{quadratic variable cost}} \tag{33}$$

where $\exp(\mu_{i,t}) - 1$ represents the expected percentage increase in productivity from R&D investment. The R&D cost function has three important components: a fixed cost, a linear variable cost, and a quadratic variable cost. It primarily affects the extensive margin by determining whether a firm undertakes R&D at all. Conditional on investing, the linear and quadratic terms capture the idea that higher R&D targets (i.e., larger $\mu_{i,t}$) require greater effort, both influencing the intensive margin of R&D investment. The quadratic component reflects how marginal costs rise disproportionately as firms pursue more ambitious R&D targets. This cost structure captures that approaching higher R&D levels requires increasingly expensive investments at the margin. To capture the dependence of R&D costs on current productivity, we allow the scale parameters (ζ_1, ζ_2) governing fixed and variable costs to vary with a firm's productivity level.

Similar to the structure of the liquidation value of tangible capital, we assume the liquidation value of intellectual capital takes a linear form, such that $\eta\Psi(\phi_{i,t}) = \eta\phi_{i,t}$.

[34] Given this specification of the R&D cost function, we can construct an alternative weight function $\omega(a, \phi) = 1 + a + \kappa\phi^{\upsilon_\phi}$ and impose the parameter restrictions $\upsilon_\phi < 2$, $\beta(1 + r) < 1$, and $\rho + \frac{\upsilon_\phi - \zeta_2}{2} < 1$. These sufficient conditions ensure that the value function $V(a, \phi)$ exists, is unique, and belongs to the weighted space $\mathcal{C}_\omega$. Specifically, the analytical expression for $\mu_{\max}(a, \phi)$ (neglecting fixed costs) is given by:

$$\mu_{\max}(a, \phi) = \ln\left(1 + \frac{\sqrt{\gamma^2 + 2\nu\frac{y(a,\phi)}{\phi^{\zeta_2}}} - \gamma}{\nu}\right).$$

Under the aforementioned parameter constraints, the following limits hold:

$$\limsup_{a\to\infty} \mathcal{R}(a, \phi) = \beta(1 + r) < 1, \quad \text{and} \quad \limsup_{\phi\to\infty} \mathcal{R}(a, \phi) = \beta\frac{C_\pi}{\kappa} < 1 \quad \text{for sufficiently large values of } \kappa,$$

where $\mathcal{R}(a, \phi) := \beta\,\frac{\sup_{I,\mu,a',c} \mathbb{E}[\omega(a',\phi')|a,\phi]}{\omega(a,\phi)}$.

5.2 Calibrated Parameters

We set the time period to be annual. There are a total of 18 parameters in the model to be determined. Instead of jointly estimate all of these parameters, we directly calibrate some parameters before estimation to simplify the estimation procedure. Specifically, we calibrate the interest rate r, capital depreciation rate δ, production function parameters α_ϕ, α_k and α_l, demand function parameter σ, utility function parameter ϵ, exogenous drift in the shock process $\bar{\mu}$, and total labor supply $\bar{L}$. These predetermined parameters are listed in Table 3.

Table 3: Calibrated Parameters Before Estimation

Parameter	Name	Source
$r = 0.06$	Interest rate	China's bank interest rate
$\delta = 0.115$	Capital depreciation rate	Median value of $\frac{\text{depreciation value}}{\text{capital stocks}}$
$\alpha_k = 0.3053$	Output elasticity of capital	Median value of $\frac{(r+\delta)k}{(r+\delta)k+wl}$
$\alpha_l = 1 - \alpha_k$	Output elasticity of labor	$1 - \alpha_k$
$\alpha_\phi = \alpha_l$	Output elasticity of productivity	Harrod-neutral technical progress
$\sigma = 2.5$	Demand elasticity	Median value of $\frac{(r+\delta)k+wl}{\text{Value Added}}$ (40% markups)
$\epsilon = 1$	Utility function parameter	As in **?**

We set the interest rate to the average benchmark deposit rate for RMB deposits at financial institutions between 2007 and 2014, as reported by the People's Bank of China, which equals 0.06.[35] The capital depreciation rate is calibrated to 0.115, corresponding to the median ratio of depreciation value to capital stock. We impose constant returns to scale, such that $\alpha_l + \alpha_k = 1$, and set the capital share to 0.3053, which represents the median value of $\frac{(r+\delta)k}{(r+\delta)k+wl}$. We assume the technology is Harrod-neutral, $\alpha_\phi = \alpha_l$.[36] The demand elasticity is set to 2.5, implying a 40% markup, which aligns with the median value of $\frac{(r+\delta)k+wl}{\text{Value Added}}$. Following **?**, we set the intertemporal elasticity of substitution ϵ to unity. We normalize mean productivity to one in the steady state without R&D by setting $\bar{\mu} = -\frac{\sigma_\xi^2}{2}$.[37] Finally, we set wage $w = 1$ and

[35] See the PBC's website http://www.pbc.gov.cn/.

[36] Since the production function is Cobb-Douglas, the distinction between Hicks-neutral and Harrod-neutral technology is purely a matter of re-scaling the technology parameter.

[37] The value of $\bar{\mu}$ is not crucial, since we can adjust other model parameters to obtain the same policy functions

ensure that $\bar{L}$ satisfies the general equilibrium condition during estimation.

5.3 Estimated Parameters

Estimation Method. We use the simulated method of moments to minimize the following objective function:[38]

$$\pounds \equiv \min_{\Theta}(M^d - M^s(\Theta))W(M^d - M^s(\Theta))'. \tag{34}$$

The vector Θ includes the key parameters summarized in Table 4. These include the discount factor β, the persistence of productivity shocks ρ_ϕ, the volatility of the random component in the shock process σ_ξ, a set of R&D cost parameters comprising the fixed cost f, linear cost γ, and quadratic cost ν, the scale parameters ζ_1 and ζ_2, the pledgeability parameters for tangible assets, θ, and the pledgeability parameters for intellectual capital before and after the policy, η_0 and η_1, respectively. M^d and $M^s(\Theta)$ denote actual data moments and simulated moments, respectively. W is a weighing matrix, computed as the inverse of the variance/covariance matrix obtained from bootstrapping the data.

We first estimate the model without the IP-backed financing policy using data from 2005-2007, by setting $\eta = \eta_0 = 0$. Since the average number of loans backed by patents is negligible before 2009 (see Figure 1), so it is reasonable to set $\eta_0 = 0$ prior to policy implementation.

Given these pre-policy parameter estimates, we then use our DID results to estimate the policy parameter $\Delta\eta$ (i.e., the change in the pledgeability of intangible induced by the policy). Hence, the pledgeability of intangible after the policy is $\eta_1 = \eta_0 + \Delta\eta = \Delta\eta$.[39]

when $\bar{\mu}$ changes.

[38]Online Appendix OA-F.1.3 provides details on the algorithms used.

[39]Since we hold all other parameters constant, our approach parallels the difference-in-differences (DID) estimation methodology. One might suggest directly using all moments from 2012-2014 to re-estimate the entire model. However, this approach would be problematic because multiple factors, policies, and shocks occurred during 2008-2011. Consequently, a new estimation would capture these confounding effects, making it impossible to isolate the impact of the patent collateral policy alone.

Table 4: Estimated Parameters

Parameter	Estimated Value	Standard Error	Name
β	0.8111	(0.0009)	Discount factor
ρ_ϕ	0.8199	(0.0042)	Persistence of $log(\phi)$
σ_ξ	1.2462	(0.0087)	Volatility of Shock
f	0.0796	(0.0178)	Fixed cost
γ	0.0006	(0.0003)	Linear cost
ν	0.9859	(0.1605)	Quadratic cost
ζ_1	0.0477	(0.0127)	Extensive margin scale
ζ_2	0.6576	(0.0070)	Intensive margin scale
θ	1.1969	(0.1165)	Pledging parameter of tangible assets
η_0	0.0000	n.a.	IP collateral parameter before policy
η_1	0.1126	(0.0078)	IP collateral parameter after policy

Data Moments. The moments were selected based on their theoretical relevance to the effect of financial constraints on R&D investment, as well as their ability to identify key structural parameters. Table 5 lists the targeted data moments.

The targeted moments include measures of RTFP dispersion—specifically the interquartile and interdecile ranges of ln(RTFP), the leverage measure captured by the median capital-to-net-worth ratio, the fraction of firms conducting R&D, and coefficients from the following empirical regressions:

$$\ln(\text{RTFP}_{i,t+1}) = \tilde{\rho}_{\text{tfp}} \ln(\text{RTFP}_{i,t}) + \tilde{\mu}_{\text{tfp}} \mathbb{1}\{\text{RDX}_{i,t} > 0\} + f_{c(i)} + f_{s(i),t} + e_{i,t}, \tag{35}$$

$$\mathbb{1}\{\text{RDX}_{i,t} > 0\} = \tilde{\beta}^{extensive}_{\text{tfp}} \ln(\text{RTFP}_{i,t}) + \tilde{\beta}^{extensive}_{\text{net worth}} \ln(\text{Net Worth}_{i,t}) + f_{c(i)} + f_{s(i),t} + e_{i,t}, \tag{36}$$

$$\ln(\text{RDX}_{i,t}) = \tilde{\beta}^{intensive}_{\text{tfp}} \ln(\text{RTFP}_{i,t}) + \tilde{\beta}^{intensive}_{\text{net worth}} \ln(\text{Net Worth}_{i,t}) + f_{c(i)} + f_{s(i),t} + e_{i,t}. \tag{37}$$

where $f_{c(i)}$ is the city-level fixed effects, capturing the city-level time-invariant characteristics, and $f_{s(i),t}$ is the industry-year fixed effects, controlling for industry-specific shocks over time. Equation (35) characterizes the profitability process. The coefficient $\tilde{\mu}_{\text{tfp}} = 0.150$ ($s.e.$ 0.0030) on the R&D indicator captures the average contribution of R&D to subsequent profitability

Table 5: Data and Model Moments

Moments	Data	Model
IQR of ln(RTFP)	1.278	1.285
IDR of ln(RTFP)	2.483	2.463
Median leverage ratio $\left(\frac{k}{\text{net worth}}\right)$	0.805	0.716
R&D Participation Rate	0.103	0.096
$\tilde{\rho}_{\text{tfp}}$	0.808	0.818
$\tilde{\mu}_{\text{tfp}}$	0.146	0.150
$\tilde{\beta}_{\text{tfp}}^{extensive}$	0.062	0.084
$\tilde{\beta}_{\text{net worth}}^{extensive}$	0.034	0.057
$\tilde{\beta}_{\text{tfp}}^{intensive}$	0.916	0.834
$\tilde{\beta}_{\text{net worth}}^{intensive}$	0.319	0.447
Change in R&D participation rate	0.0245	0.0245

Note: The last row of the table, "Change in R&D participation rate", is the average effect of the policy on the fraction of firms conducting R&D, as estimated from the DID regression in Model 1 of Table 1. This moment is used to identify the policy parameter η_1 after the policy.

(RTFP). It indicates that, conditional on current RTFP, firms with strictly positive R&D expenditure exhibit approximately 15% higher RTFP in the following period, compared with firms without R&D investment. Financial constraints inhibit R&D investment, so tighter financial constraints could induce a smaller $\tilde{\mu}_{\text{tfp}}$. Moreover, the productivity increment $\tilde{\mu}_{\text{tfp}}$ is also related to R&D costs, with higher costs implying lower R&D investment and smaller $\tilde{\mu}_{\text{tfp}}$. The parameter $\tilde{\rho}_{\text{tfp}} = 0.818$ (*s.e.* 0.0017) measures the persistence of productivity.

Equation (36) proxies the policy function for the R&D participation decision. The estimated coefficients $\tilde{\beta}_{\text{tfp}}^{extensive} = 0.062$ (*s.e.* 0.0008) and $\tilde{\beta}_{\text{net worth}}^{extensive} = 0.034$ (*s.e.* 0.0004), indicate that both higher profitability and greater net worth are associated with an increased probability of conducting R&D. This finding is consistent with our model predictions.

Equation (37) approximates the policy function for the level of R&D expenditure. The coefficients $\tilde{\beta}_{\text{tfp}}^{intensive} = 0.916$ (*s.e.* 0.0135) and $\tilde{\beta}_{\text{net worth}}^{intensive} = 0.319$ (*s.e.* 0.0072) reveal that, conditional on positive R&D expenditure, the elasticity of R&D spending with respect to

RTFP is 0.916, while the elasticity with respect to net worth is 0.319. These estimates indicate that, on average, R&D expenditures among active firms increase with both productivity and net worth. In our theory, the former relationship is state-dependent and thus governed by the distribution of firms, whereas the latter relationship is unambiguous and aligns with this regression result.

In short, these regression results show two key relationships. First, due to financial constraints, R&D investment depends on firm net worth, as evidenced by the positive coefficients $\tilde{\beta}^{extensive}_{\text{net worth}}$ and $\tilde{\beta}^{intensive}_{\text{net worth}}$. Second, higher productivity firms are more likely to engage in R&D and invest more intensively, reflected in the positive values of $\tilde{\beta}^{extensive}_{\text{tfp}}$ and $\tilde{\beta}^{intensive}_{\text{tfp}}$.

The last moment is the result of the DID regression from Model 1 in Table 1, reflecting the average effect of the policy on the fraction of R&D four-to-six years after the policy, that is, the fraction of firms having strictly positive R&D increased by 2.45%.

We noticed that our DID results have two key features: (1) they show the policy effect four-to-six years after implementation; (2) they capture the partial equilibrium effect of the policy. Given these features of the DID analysis, when estimating the policy parameter, (1) we focus on the transition path in our estimation and calculate the average policy effect four-to-six years after implementation; and (2) we use only partial equilibrium (PE) results.[40] We return to the general equilibrium model when calculating policy effects under counterfactual scenarios where the policy is implemented at the nationwide level.

5.4 Identification

The model is both nonlinear and rich in parameters and moments. There is no direct one-to-one mapping between parameters and the moments that identify them. However, certain moments relate more intuitively and closely to specific parameters. We describe these relationships below.

The dispersion measures (interquartile and interdecile ranges) of RTFP capture firm productivity heterogeneity and are positively related to the standard deviation of the random component in the shock process, σ_ξ. Due to endogenous R&D choices, this dispersion also

[40] Using general equilibrium results in the estimation would yield a significantly larger estimated value of $\Delta\eta$ than our current approach.

reflects heterogeneous R&D investment decisions across firms.

The median capital-to-net-worth ratio reflects entrepreneurs' savings incentives to avoid future financial constraints. The savings motivation is closely tied to the discount factor β. Higher patience (larger β) increases savings incentives, leading to greater net worth accumulation and a lower capital-to-net-worth ratio. It is also related to θ, which reflects the pledgeability of tangible assets. In contrast to β, an increase in θ induces a rise in the capital-to-net-worth ratio by enabling firms to obtain additional financing for capital investment.

The fraction of firms conducting R&D captures the extensive margin of R&D activity and is primarily related to R&D costs, particularly the fixed cost f. Regression (35) mimics the productivity shock process (6), so higher persistence ρ_ϕ implies a higher estimated coefficient $\tilde{\rho}_{\text{tfp}}$. The coefficient $\tilde{\mu}_{\text{tfp}}$ captures the average impact of R&D on productivity for firms with positive R&D expenditure, making it sensitive to the linear and quadratic cost parameters, γ and ν.

The scale parameters ζ_1 and ζ_2 are identified through the results of two regressions (36) and (37). A larger ζ_1 increases R&D startup costs for high-productivity firms, reducing the probability of R&D engagement among these firms. Hence, $\tilde{\beta}_{\text{tfp}}^{extensive}$ decreases with ζ_1. Similarly, a larger ζ_2 increases the marginal cost of R&D intensity for high-productivity firms, reducing R&D investment levels. Hence, $\tilde{\beta}_{\text{tfp}}^{intensive}$ decreases with ζ_2.

Finally, the identification of the policy parameter η_1 uses the fact that firms adjust their R&D investment after the promotion of IP-backed financing. Proposition 3 and numerical simulations confirm that promoting IP-backed financing stimulates R&D investment. Given other parameters, an increase in η raises the marginal benefit of R&D and relaxes financial constraints, both of which encourage R&D investment. We select a conservative moment to match—the extensive margin effect of the policy, specifically the increase in the firm's R&D participation rate.[41]

Table A.2 in Appendix F reports the elasticities of moments with respect to parameters, revealing how each moment responds locally to changes in individual parameters. The signs of these elasticities are consistent with the discussion above. In addition, in Table 4, it is quite

[41] An alternative choice would be to target moments of the change in $\ln(\text{RDX}_{i,t} + 1)$, which would yield a larger estimate of η_{after} after the policy.

clear that all parameter estimates (except for the linear cost) are very precise as the standard errors are quite small.

5.5 Estimation Results

Table 4 presents the baseline estimation results.

Parameters for the Borrowing Constraint. The estimated pledgeability of tangible assets $\theta = 1.2$ slightly exceeds Moll's estimate of 1.06 for China (in year 1997) and matches India's value of 1.2 (in year 1997), but remains well below the US estimate of 4.15 (in year 1997). The estimated pledgeability of intellectual capital after the policy, $\eta_1 = 0.1126$. Although direct empirical counterparts for this parameter are unavailable, Figure 6 illustrates that the estimated change in η successfully accounts for both the shift in the R&D participation rate (a targeted moment) and the average change in R&D intensity (an untargeted moment).

Productivity Persistence and Volatility. The productivity persistence parameter $\rho_\phi = 0.82$ implies that R&D contributions to log productivity have a half-life of approximately 3.5 years ($\log(0.5)/\log(0.82)$). The volatility of innovation shocks $\sigma_\xi = 1.25$ indicates substantial variation in intellectual capital quality (or productivity) across time.

R&D Cost Structure. Since the cost parameters are unit-free, we examine the mean incurred R&D expenditures relative to value added among R&D-conducting firms. Total incurred R&D expenditure averages 18% of value added, closely matching our full sample estimate of 16%. Decomposing by cost type, fixed costs account for 5.7% of value added, linear costs are negligible (near zero), and quadratic costs represent 12%. The scale parameters ζ_1 and ζ_2 are both significantly positive, confirming that high-productivity firms face higher R&D costs to achieve equivalent growth rates as low-productivity firms.

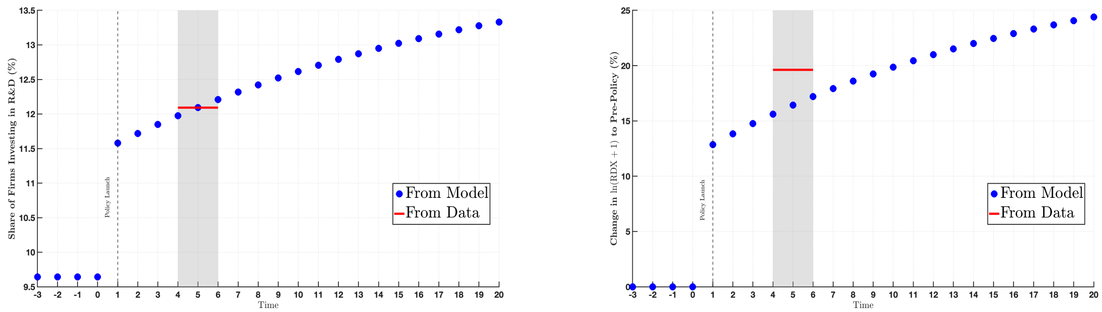
Discount Factor. The estimated discount factor, $\beta = 0.81$, is notably lower than the standard range of 0.90–0.95 commonly used in the literature, where β is typically calibrated rather than estimated. We interpret this low discount factor as capturing additional frictions beyond borrowing constraints—such as input (capital or labor) adjustment costs or policy distortions—that affect firms' future valuations but are not explicitly modeled.

5.6 Model Validity

Model Fit. Table 5 presents both the targeted data moments and their model-generated counterparts. Overall, the model successfully replicates the key features of the data. It accurately captures RTFP dispersion as measured by the interquartile and interdecile ranges. The median capital-to-net-worth ratio is marginally underestimated, implying that firms in the model accumulate slightly more net worth than observed in the data. The R&D participation rate is also matched well. Furthermore, the estimated productivity persistence and the average return to R&D closely track the empirical moments. Although the simulation implies a somewhat higher sensitivity of R&D choices to net worth than the empirical estimates in equations (36) and (37), the model generally captures the key empirical patterns, validating its use for subsequent policy analysis.

The model also successfully reproduces the observed increase in R&D participation following the IP-backed financing policy, matching the 2.45% rise documented in our DID analysis (Figure 6a). Crucially, the model performs well on non-targeted moments: it generates an average increase of 16.4% in R&D expenditure (measured as $\ln(\text{R\&D} + 1)$) four to six years post-policy, closely aligning with the empirical estimate of 19.6% (Figure 6b). This alignment between the model and the data validates the model's ability to capture firms' R&D responses to policy-induced shifts in financial constraints.

Finally, the model qualitatively reproduces the policy's positive impact on firm RTFP and the reduction in MRPK dispersion (Figure A.4 in Appendix H.1). Quantitatively, the model yields a conservative estimate of the RTFP increase relative to the data. This difference likely reflects the model's exclusive focus on the financial constraint channel via changes in IP pledgeability; by design, it abstracts from other mechanisms—such as direct subsidies, enhanced innovation efficiency and quality, or regional knowledge spillovers—that may further boost productivity in the data. Conversely, the model predicts a sharper decline in MRPK dispersion than observed empirically. This is expected, as the model abstracts from frictions such as adjustment costs or information asymmetries, which typically dampen the capital reallocation. Furthermore, it is important to note that the dispersion of MRPK is an imperfect proxy for true allocative efficiency, both in the model and in the data. Despite these quantitative differences, the model's ability to replicate the directional shifts in these non-targeted moments validates the core economic mechanism.



(a) Average $\mathbb{1}\{R\&D > 0\}$

(b) Change in $\ln(RDX + 1)$

Figure 6: Policy Effects on R&D Activities: Model vs. Data

Note: The shaded area represents the simulation's counterpart to the empirical data period (2012–2014), with red solid dots representing the coefficient β_1 of $\text{Pilot}_i \times \text{After}_t$ in the DID regression (1) using the actual data.

Characterization of the Borrowing Constraints. As shown in Proposition 1, the model implies that firms face tighter financial constraints have higher ln(MRPK). To examine the model's validity for characterizing the borrowing constraints, we run the following regression using both actual data and simulated data from our baseline model before policy implementation:

$$\ln(\text{MRPK}_{i,t}) = \beta_0 + \beta_1 \ln(\text{Net Worth}_{i,t}) + \beta_2 \ln(\text{RTFP}_{i,t}) + f_{c(i)} + f_{s(i),t} + e_{i,t} \tag{38}$$

Table 6 reports the coefficients for ln(Net Worth) and ln(RTFP). In the data, there is a significant negative relationship between MRPK and net worth, and a significant positive relationship between MRPK and the productivity. These are consistent with the main result in Proposition 1. Moreover, the estimated model characterizes these relationships quite well, demonstrating that firms with lower net worth or higher productivity face tighter financial constraints.

Table 6: Relationship between MRPK, Net Worth, and RTFP

	Data		Model
	Model 1	Model 2	Model 3
Dependent Variable	ln(MRPK)	ln(MRPK)	ln(MRPK)
ln(Net Worth)	-0.4142***	-0.4037***	-0.2262***
	(0.0013)	(0.0013)	(0.0002)
ln(RTFP)	0.6417***	0.6347***	0.5585***
	(0.0024)	(0.0024)	(0.0005)
Observations	470,241	470,241	1,200,000
R-squared	0.318	0.332	0.578
City FE	Yes	Yes	–
Time FE	Yes	No	–
Industry FE	Yes	No	–
Time-Industry FE	No	Yes	–

Note: Robust standard errors are provided in parentheses.
*** p<0.01, ** p<0.05, * p<0.1

6 Counterfactual Analysis

Although our DID estimates show that the pilot program affects firm-level innovation, productivity, and the allocation of capital across firms, reduced-form evidence alone cannot translate these effects into aggregate consequences, characterize the transition dynamics, or disentangle the relative contributions of within-firm innovation and across-firm reallocation. Nor can it support disciplined counterfactual exercises—such as a nationwide rollout or comparisons to policies that expand lending against tangible collateral—because these analyses require general equilibrium feedback and policy-invariant primitives. We therefore use the estimated general equilibrium model to quantify the policy's aggregate and dynamic implications and perform these counterfactual exercises.

6.1 Aggregate TFP Effects of Nationwide IP-backed financing

The first counterfactual experiment we consider is implementing the IP-backed financing policy nationwide. Specifically, we impose $\eta = 0.1126$ and compute the general equilibrium results.[42]

Figure 7 shows that the transition dynamics of the aggregate TFP gains from static and dynamic channels. Specifically, when the policy is implemented economy-wide, ATFP increases by approximately 14% in the long run, with dynamic TFP gains reaching around 4.4% and the rest 9.6% arises from improved capital misallocation.

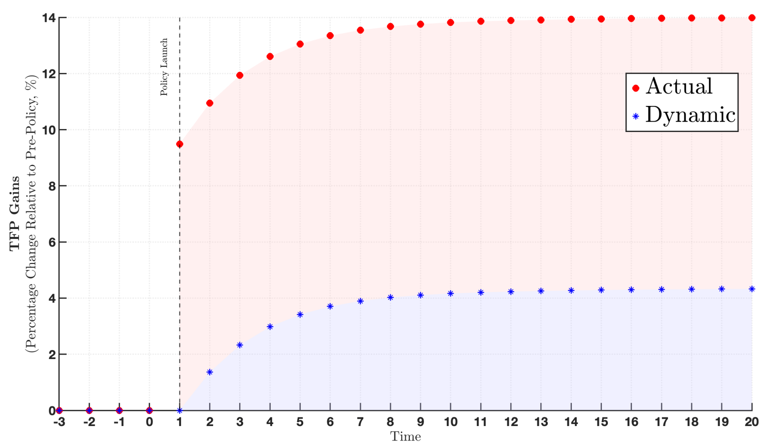


Figure 7: Productivity Gains from Nationwide IP-backed financing in General Equilibrium
Note: We calculate the percentage change for each variable relative to its corresponding value prior to the policy implementation.

We also compute the partial-equilibrium results under fixed wages, reported in Appendix G. Relative to this fixed-wage benchmark, long-run productivity gains are substantially smaller in general equilibrium (14% in GE vs. 35% in PE).[43] This gap is driven primarily by the dynamic component of productivity.[44] In partial equilibrium, long-run dynamic gains are about 25%,

[42]We allow wages to adjust to clear the labor market and solve for the general equilibrium using a shooting method. In this approach, we guess an initial wage path, solve the firm's problem and market clearing conditions backward to generate a new wage path, and iterate until the wage path converges. Online Appendix OA-F.1.2 provides more details.

[43]See Figure A.3a in Appendix G.

[44]The transition path of the standard deviation of ln(MRPK) exhibits no significant difference from that under partial equilibrium, indicating that general equilibrium effects have negligible impact on allocation efficiency improvements.

accounting for roughly 70% of the total productivity increase. In general equilibrium, by contrast, dynamic gains account for only about 30% of the total. The mechanism is that wages rise endogenously in general equilibrium, which reduces the marginal return to R&D and dampens R&D investment. This highlights the importance of incorporating general-equilibrium price effects when assessing R&D decisions and their aggregate implications.[45]

6.2 Comparison with Tangible-Backed Lending Reform

In this section, we contrast the IP-backed financing policy with a conventional tangible-backed lending. Absent intangibles, the standard collateral constraint is $k_{i,t} \leq \theta a_{i,t}$ (e.g., **?**). While in our model, borrowing constraint (9) implies that increasing η relaxes the constraint through increasing the pledgeability of intellectual capital. A direct comparison is not straightforward because the IP reform implies a firm-specific "effective collateral multiplier." To see it, rewrite the constraint as

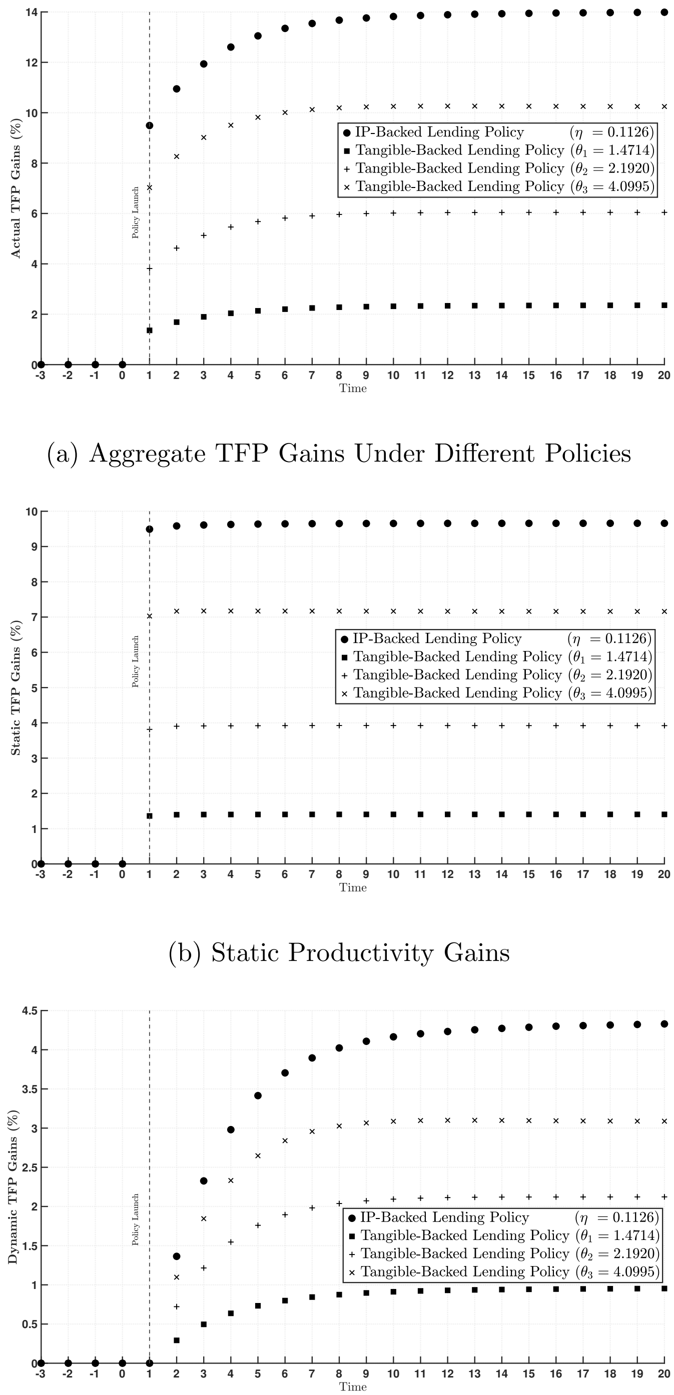
$$k_{i,t} \leq \kappa_{i,t}\, a_{i,t}, \qquad \kappa_{i,t} \equiv \theta + \eta \frac{\phi_{i,t}}{a_{i,t}},$$

which varies across firms through $\phi_{i,t}/a_{i,t}$. A single θ therefore cannot replicate the full post-reform cross-sectional distribution of $\kappa_{i,t}$. Instead, we discipline tangible-collateral counterfactuals by matching percentiles of κ_i in the post-policy GE steady state under the IP policy, and then set $\eta = 0$. Specifically, we consider three values: a median match $\theta_{50} \equiv P_{50}(\kappa_i) = 1.4714$, an upper-quartile match $\theta_{75} \equiv P_{75}(\kappa_i) = 2.1920$, and an upper-decile match $\theta_{90} \equiv P_{90}(\kappa_i) = 4.0995$ (close to the U.S. estimate in **?**). For each new θ_j ($j \in \{50, 75, 90\}$), we solve the GE transition from the pre-policy steady state to the new steady state and compute ATFP, ETFP, and allocation efficiency.

Figure 8a shows that even large tangible-collateral expansions cannot match the aggregate TFP gains from IP-backed financing: long-run TFP gains rise with θ (from below 2.5% at θ_{50} to about 10% at θ_{90}) but remain below the IP policy.[46] The decomposition clarifies why: even at θ_{90}, both the allocation-efficiency improvement (about 7% vs. 10%) and the effective-TFP gain (about 3% vs. 4.5%) fall short (see Figures 8b–8c).

[45] Figures A.7a and A.7b in Appendix H.2 show that the increases in both the extensive and intensive margins of R&D investment are smaller than in partial equilibrium.

[46] Matching the IP policy's long-run TFP gain with a tangible-collateral reform alone would require an implausibly large $\theta \approx 20.7$, far above the U.S. benchmark.



(a) Aggregate TFP Gains Under Different Policies

(b) Static Productivity Gains

(c) Dynamic Productivity Gains

Figure 8: Comparison between IP-backed financing and Tangible-Backed Lending Policies

Note: We calculate the percentage change for each variable relative to its corresponding value prior to the policy implementation.

As shown in Proposition 1 and confirmed in both the data and estimated model (see Table 6), firms with lower net worth or higher productivity tend to face tighter financial constraints. For firms with low net worth and high productivity which need more borrowing, traditional tangible-backed lending policy has limited impact. In contrast, the IP-backed financing is more effective at alleviating financial constraints for those facing the most severe credit limits. This difference makes the IP-backed financing policy more effective at alleviating capital misallocation.

Furthermore, as discussed in subsection 4.4, compared to the tangible-backed lending policy, the IP-backed financing policy stimulates R&D investment more effectively for firms seeking to avoid binding constraints in the future (particularly firms with relatively high productivity and low net worth; see Figure 5). This is because increasing η directly affects the marginal benefit of R&D investment by enhancing the pledgeability of intellectual capital, thereby relaxing firms' future borrowing constraints. In contrast, tangible-backed lending does not offer this direct mechanism. This explains why IP-backed financing is more efficient at improving firm-level productivity, leading to larger dynamic gains.

6.3 Extensions

Long-Run Growth Effects. We also calibrate the extended growth model with R&D spillovers outlined in Subsection 4.6. Online Appendix OA-G.3 presents the calibration procedure. Given these parameters, we examine both the IP-backed financing policy's impact on allocative efficiency at the balanced growth path (a level effect) and the growth rate of aggregate productivity. Our results demonstrate that capital's allocative efficiency improves by 4.43% on the new balanced growth path following the policy intervention, and the long-run growth rate of aggregate productivity increases by 0.15 percentage points.

Exogenous Productivity Process without R&D Investment. To further explore the role of R&D investment in driving productivity improvements, we also consider an alternative model specification where the productivity process is exogenous. In this model, firms do not endogenously choose R&D investment, and only choose net worth. Therefore, the policy only affects firms through relaxing their borrowing constraints, without directly influencing their incentives to invest in R&D through enhancing the collateral value of intellectual capital. We shut down the R&D investment channel by setting the firm's R&D target $\mu_{i,t}$ to be constant and equal to the mean value in the baseline model before the policy change. We then solve for the general equilibrium transition and compute the aggregate TFP gains from the policy. The transition paths are shown in Figure OA-6a for the change in η and Figure OA-6b for the change in θ in Online Appendix OA-H.1. First, it is unsurprising that dynamic gains are zero given the absence of R&D investment; consequently, the divergence in policy effects between this model and the baseline captures the dynamic gains. Second, even in an environment without R&D choices, IP-backed financing outperforms tangible-backed lending. This mirrors the baseline findings: IP-backed financing alleviates financial constraints for high-productivity, low-net-worth firms—precisely those facing the most severe credit limits.

7 Conclusion

This paper studies how expanding the pledgeability of intellectual property affects innovation and aggregate productivity. Motivated by the growing role of intangibles in modern production and the continued reliance of credit markets on tangible collateral, we investigate China's patent pledge financing pilot program, which introduced institutional and fiscal support for lending against patents in selected cities beginning in 2009.

Using the pilot as a quasi-natural experiment and combining policy information with administrative firm-level data, we document three robust facts. First, access to IP-backed financing increases innovative investment: four to six years after implementation, the probability that a firm conducts R&D rises by 2.45 percentage points, and R&D spending increases along both the extensive and intensive margins. Second, firm performance improves: revenue-based productivity increases by about 0.29 log points in treated cities relative to controls. Third, capital allocation becomes more efficient: dispersion in MRPK at the city–industry level declines sharply, consistent with relaxed collateral constraints reducing misallocation.

To translate these reduced-form estimates into aggregate implications and to isolate the mechanisms through which IP pledgeability operates, we build and estimate a heterogeneous-firm general equilibrium model with collateralized borrowing constraints backed by both tangible and intangible assets. A key ingredient is that R&D not only raises future productivity but also expands the stock of future pledgeable intangible collateral, strengthening incentives to innovate when intangibles become financeable. We estimate the model using the method of simulated moments on pre-policy data and identify the reform-induced change in intangible pledgeability by matching the DID estimate of the policy effect on R&D participation.

The estimated model implies large aggregate gains from scaling the reform. A nationwide adoption of IP-backed financing raises long-run aggregate TFP by about 14 percent. Importantly, these gains are not purely static: roughly 10 percentage points reflect improved allocative efficiency from better capital reallocation, while about 4 percentage points reflect dynamic innovation responses that raise effective productivity over time. This decomposition underscores that treating IP-backed financing as simply another credit expansion misses a central source of long-run gains: when innovation is endogenous, reforms that make intangible assets pledgeable change the dynamic return to R&D.

The model provides an economic interpretation for why the empirical effects are large and why they persist beyond the immediate post-reform period. Allowing IP to enter the collateral constraint lowers the shadow cost of external finance for constrained firms and compresses cross-sectional dispersion in marginal products, generating immediate static TFP gains through improved allocation. At the same time, innovation responds not only because current financing conditions improve, but because higher IP pledgeability raises the dynamic return to building intellectual capital: when a larger share of future intellectual capital becomes collateralizable, R&D delivers an additional payoff by relaxing tomorrow's borrowing constraint. This feedback loop from innovation to financing capacity and back to innovation is absent in standard environments with exogenous productivity or with collateral restricted to tangible assets.

We also compare IP-backed financing to conventional tangible-collateral reforms. Even sizable increases in tangible pledgeability generate smaller productivity gains than IP-backed financing. The difference is both distributional and mechanistic. A uniform relaxation of tangible-collateral constraints primarily scales borrowing capacity with existing net worth and therefore tends to favor firms that already hold pledgeable wealth. By contrast, higher intangible pledgeability ties borrowing capacity more directly to productivity-relevant assets, disproportionately relaxing constraints for high-productivity, low-wealth firms—those with high marginal products of capital and strong incentives to expand and innovate—and it activates a collateral-creation motive that further amplifies R&D investment. In this sense, intangible-collateral reforms can be more targeted and more growth-relevant than broad expansions of tangible-backed credit, and accounting for both static reallocation gains and dynamic innovation gains is essential for evaluating their long-run impact.

Several limitations point to directions for future work. First, our data do not include bank–firm loan contracts or detailed information on collateral valuation and recovery, which precludes a clean decomposition of which institutional features—such as valuation standards, enforcement, subsidies, and risk-sharing arrangements—account for the observed increase in pledgeability. Second, our baseline model focuses on level effects on productivity. While we provide a simple extension with knowledge spillovers to illustrate the potential for longer-run growth effects, richer data would allow more credible identification of spillovers and growth responses to innovation. Third, improved measures of patent quality and the depth and liquidity of secondary markets for IP would help link policy-induced expansions in financing to the underlying economic value of pledged intangible assets.

Appendix

A Robustness Check — Balanced Panel with Firm Fixed Effects

The baseline results reported in Table 1 rely on two datasets with disparate ID systems, making it difficult to track firms over time. To address this, we restrict the following exercise to tax data (SAT) and utilize a balanced panel. We define 2011 as the pre-policy period and 2014 as the post-policy period. All other variable definitions remain consistent with the baseline. We re-estimate the models controlling for firm fixed effects, and the results are reported in Table A.1.

Table A.1: The Effects of Policy: Controlling Firm Fixed Effects

	Model 1	Model 2	Model 3	Model 4
Dependent Variable	$\mathbb{I}_{\text{RDX}>0}$	$\ln(\text{RDX}+1)$	$\ln(\text{RTFP})$	$\sigma^{\text{mrpk}}_{c,s,t}$
Pilot · After	0.0141**	0.1098**	0.0326**	-0.1300**
	(0.0068)	(0.0540)	(0.0152)	(0.0600)
Observations	135,946	135,946	73,105	16,251
R-squared	0.658	0.695	0.928	0.153
Firm FE	Yes	Yes	Yes	-
City FE	No	No	No	Yes
Industry-Year FE	Yes	Yes	Yes	Yes

Note: Robust standard deviations are provided in parentheses.
*** p<0.01, ** p<0.05, * p<0.1

B Model Derivation Details

B.1 Demand Function

Although we do not consider international trade explicitly, we can conceptualize a world market where competitive final goods producers operate with the production function

$$Y_w = \left(Y_d^{\frac{\sigma-1}{\sigma}} + Y_f^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

where Y_w denotes world final goods, Y_d represents the bundle of intermediate goods from domestic producers, and Y_f represents the bundle of intermediate goods from foreign countries. These bundled intermediate goods take the following functional form:

$$Y_j = \left(\int_{i \in N_j} q_i^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}$$

where $j \in \{d, f\}$ and N_j denotes the set of firms in region j. The demand function for domestic good i is then given by

$$q_i = p_i^{-\sigma} \frac{Y_w}{P_w^{1-\sigma}}$$

where $P_w = \left(\int_{i \in N_d} p_i^{1-\sigma} di + \int_{i \in N_f} p_i^{1-\sigma} di \right)^{\frac{1}{1-\sigma}}$ is the price index of final good Y_w. Thus, the demand shifter is $D = \frac{Y_w}{P_w^{1-\sigma}}$ and can be treated as exogenous in the small open economy.

B.2 Borrowing Constraint

We provide a formal derivation of the borrowing constraint function. Let $V_t^D(a, \phi)$ be the value function when the firm defaults and $V_t^N(a, \phi)$ be the value function when the firm does not default. When not defaulting, the value function is given by

$$V_t^N(a, \phi) = V_t(a, \phi) = \max_{b', k', \mu} \left\{ U(c) + \beta \mathbb{E}_t V_{t+1}(a', \phi') \right\} \tag{39}$$

subject to the constraint:

$$c + \underbrace{k' - b'}_{=a'} + \chi(\mu; \phi) = \pi(a, \phi; w_t) - (1 + r)b$$

where b is the amount of debt in the next period, and k' is the physical capital in the next period. Because of limited enforcement of contracts, a firm can default on a fraction of the face value of current debt. Therefore, the firm only needs to pay $(1+r-\mu_0)b$ to the bank. The cost of defaulting is a fraction of collateral used when borrowing from the bank: $\mu_1(1-\delta)k+\mu_2\Psi(\phi)$, $\mu_1, \mu_2 \in (0, 1)$. In other words, the bank can seize a fraction of the capital and the value of intellectual capital when the firm defaults. We assume that the firms only default for one period and have access to the financial market in the next period. This implies that $V^D(a, \phi)$ can be expressed as:

$$V_t^D(a, \phi) = \max_{b', k', \mu} \left\{ U(c) + \beta \mathbb{E}_t V_{t+1}(a', \phi') \right\} \tag{40}$$

subject to the constraint:

$$c + a' + \mu_1(1-\delta)k + \mu_2\Psi(\phi) + \chi(\mu;\phi) = \pi(a,\phi;w_t) - (1+r-\mu_0)b$$

The condition for an equilibrium in which no firm defaults is $V_t^D(a,\phi) \le V_t^N(a,\phi)$, which implies that

$$(1+r)b \le (1+r-\mu_0)b + \mu_1(1-\delta)k + \mu_2\Psi(\phi) \tag{41}$$

This immediately implies that

$$b \le \frac{\mu_1(1-\delta)}{\mu_0}k + \frac{\mu_2}{\mu_0}\Psi(\phi) \tag{42}$$

Define that $\tilde{\theta} = \frac{\mu_1(1-\delta)}{\mu_0}$ and $\tilde{\eta} = \frac{\mu_2}{\mu_0}$, we have the borrowing constraint as in the main text. To make the capital constraint meaningful, we restrict that $\tilde{\theta} \in (0, 1)$.

C Proofs

In this section, we provide proof sketches for the key equations and propositions. Detailed proofs can be found in Section OA-D of the online appendix.

C.1 Proofs of Equation (16), (17), and (18)

Proofs of Equation (16), (17), *and* (18). The firm's optimization problem is

$$\max_{l,k} \left\{ (\phi^{\alpha_\phi} k^{\alpha_k} l^{\alpha_l})^{\frac{\sigma-1}{\sigma}} - wl - (r+\delta)k \right\}$$
$$s.t.\ k \le \theta a + \eta\Psi(\phi).$$

Using the scaled output elasticities $\tilde{\alpha}_\ell = \frac{\sigma-1}{\sigma}\alpha_\ell$, the revenue function is $R = \phi^{\tilde{\alpha}_\phi} k^{\tilde{\alpha}_k} l^{\tilde{\alpha}_l}$. Let λ denote the Lagrange multiplier on the capital constraint. The Lagrangian is:

$$\mathcal{L} = \phi^{\tilde{\alpha}_\phi} k^{\tilde{\alpha}_k} l^{\tilde{\alpha}_l} - wl - (r+\delta)k + \lambda(\theta a + \eta\Psi(\phi) - k)$$

The first-order conditions (FOCs) for labor and capital are:

$$\frac{\partial \mathcal{L}}{\partial l} = \tilde{\alpha}_l \frac{R}{l} - w = 0 \implies l = \frac{\tilde{\alpha}_l R}{w}, \tag{43}$$

$$\frac{\partial \mathcal{L}}{\partial k} = \tilde{\alpha}_k \frac{R}{k} - (r + \delta + \lambda) = 0. \tag{44}$$

Let $M \equiv r+\delta+\lambda$ denote the shadow price of capital. From (44), we have $R = \frac{Mk}{\tilde{\alpha}_k}$. Substituting this into (43) gives the optimal labor-capital ratio:

$$l = \frac{\tilde{\alpha}_l}{\tilde{\alpha}_k w} Mk. \tag{45}$$

Now, substitute (45) back into the revenue function expression $R = \frac{Mk}{\tilde{\alpha}_k}$:

$$\frac{Mk}{\tilde{\alpha}_k} = \phi^{\tilde{\alpha}_\phi} k^{\tilde{\alpha}_k} \left(\frac{\tilde{\alpha}_l Mk}{\tilde{\alpha}_k w} \right)^{\tilde{\alpha}_l}$$

Solving for k yields the optimal capital demand as a function of the shadow price M:

$$k^* = \left(\frac{\tilde{\alpha}_k^{1-\tilde{\alpha}_l} \tilde{\alpha}_l^{\tilde{\alpha}_l}}{w^{\tilde{\alpha}_l}} \right)^{\frac{1}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} M^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} = \Gamma(w) M^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}. \tag{46}$$

This matches Equation (17). Substituting k^* into (45) gives l^*:

$$l^* = \frac{\tilde{\alpha}_l}{\tilde{\alpha}_k w} \Gamma(w) M^{-\frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}.$$

This matches Equation (16).

To find the shadow price M, we consider the constraint. If the constraint is not binding ($\lambda = 0$), then $M = r + \delta$. If the constraint is binding ($\lambda > 0$), then $k = \theta a + \eta \Psi(\phi)$. We invert the $k^*(M)$ equation:

$$M = \left(\frac{\Gamma(w) \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}}{k} \right)^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}} = \frac{\Gamma(w)^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}} \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_l}}}{k^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}}}.$$

Substituting the binding capital limit $k = \theta a + \eta \Psi(\phi)$, we obtain the expression in (18). $\square$

See more details in Section OA-D.2 of the online appendix.

C.2 Proof of Proposition 1

Proof of Proposition 1. Let $k^u(\phi)$ denote the unconstrained capital demand, which corresponds to the shadow price $M = r + \delta$. From Equation (46), we have $k^u(\phi) \propto \phi^{\upsilon_\phi}$ with $\upsilon_\phi := \frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}$, implying $\frac{d \log k^u(\phi)}{d \log \phi} = \upsilon_\phi$.

We first establish and subsequently examine the properties of the threshold $\hat{a}(\phi)$ the properties of $M(a, \phi)$ in the binding region (i.e., where $a < \hat{a}(\phi)$).

Proposition of $\hat{a}(\phi)$. The threshold $\hat{a}(\phi)$ is defined as the net worth where the firm's borrowing capacity exactly equals this unconstrained demand:

$$k^u(\phi) = \theta\hat{a}(\phi) + \eta\Psi(\phi). \tag{47}$$

Let $\varsigma_{\hat{a}(\phi)} := \frac{d\log\hat{a}(\phi)}{d\log\phi}$ and $\varsigma_\Psi(\phi) := \frac{d\log\Psi(\phi)}{d\log\phi}$. To find the elasticity $\varsigma_{\hat{a}(\phi)}$, we differentiate (47) with respect to ϕ:

$$\frac{dk^u(\phi)}{d\phi} = \theta\frac{d\hat{a}(\phi)}{d\phi} + \eta\frac{d\Psi(\phi)}{d\phi}.$$

Multiplying the equation by ϕ allows us to express terms in elasticities (using the identity $x\frac{df}{dx} = f \cdot \frac{d\log f}{d\log x}$):

$$k^u(\phi)\upsilon_\phi = \theta\hat{a}(\phi)\varsigma_{\hat{a}(\phi)} + \eta\Psi(\phi)\varsigma_\Psi(\phi).$$

Solving for $\varsigma_{\hat{a}(\phi)}$:

$$\varsigma_{\hat{a}(\phi)} = \frac{k^u(\phi)\upsilon_\phi - \eta\Psi(\phi)\varsigma_\Psi(\phi)}{\theta\hat{a}(\phi)}. \tag{48}$$

Factoring out $\frac{k^u(\phi)}{\theta\hat{a}(\phi)}$ from (48) yields (20).

To determine the sign, we return to equation (48). Using the binding condition $\theta\hat{a}(\phi) = k^u(\phi) - \eta\Psi(\phi)$, we can rewrite the elasticity as:

$$\varsigma_{\hat{a}(\phi)} = \frac{k^u(\phi)\upsilon_\phi - \eta\Psi(\phi)\varsigma_\Psi(\phi)}{k^u(\phi) - \eta\Psi(\phi)}.$$

For the threshold to be economically meaningful (i.e., $\hat{a}(\phi) > 0$), the unconstrained capital must exceed the collateral value, $k^u(\phi) > \eta\Psi(\phi)$, ensuring the denominator is positive.

Under Assumption 2, we have $\upsilon_\phi > \varsigma_\Psi(\phi)$ for all ϕ. Since $k^u(\phi) > \eta\Psi(\phi) > 0$, it follows that:

$$k^u(\phi)\upsilon_\phi > k^u(\phi)\varsigma_\Psi(\phi) > \eta\Psi(\phi)\varsigma_\Psi(\phi).$$

Thus, the numerator is strictly positive, implying $\varsigma_{\hat{a}(\phi)} > 0$.

Properties of $M(a,\phi)$. In the region where $a < \hat{a}(\phi)$, Equation (18) implies:

$$M(a,\phi) = \frac{\Gamma(w)^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}l}}\phi^{\frac{\tilde{\alpha}\phi}{1-\tilde{\alpha}_l}}}{[\theta a + \eta\Psi(\phi)]^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}}}. \tag{49}$$

An increase in a raises the denominator while leaving the numerator unchanged; thus, $M(a,\phi)$ is strictly decreasing in a within the binding region.

To analyze the sensitivity of $M(a, \phi)$ with respect to ϕ, we take the natural logarithm of Equation (49) and compute the partial derivative $\frac{\partial \ln M}{\partial \ln \phi}$:

$$\begin{aligned}\frac{\partial \ln M(a,\phi)}{\partial \ln \phi} &= \frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}\left(\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l} - \frac{\eta\Psi(\phi)}{\theta a+\eta\Psi(\phi)}\frac{\partial \ln \Psi(\phi)}{\partial \ln \phi}\right) \\ &\geq \frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}\left(\frac{d\log k^u(\phi)}{d\log \phi} - \frac{\partial \ln \Psi(\phi)}{\partial \ln \phi}\right) > 0,\end{aligned}$$

where the inequality holds because $0 \leq \frac{\eta\Psi(\phi)}{\theta a+\eta\Psi(\phi)} < 1$ and the final strict inequality follows from Assumption 2. $\square$

See more details in Section OA-D.3 of the online appendix.

C.3 Proof of Proposition 2

We first establish the following lemmas, which are used in the proof of Proposition 2. The proofs of these lemmas are deferred to Online Appendix OA-D.4.1.

Let $I \in \{0,1\}$ denote the R&D investment indicator, where $I=1$ if firms invest in R&D, and $I=0$ otherwise.

Lemma 5. Given the law of motion for productivity

$$\ln(\phi') = \rho\ln(\phi) + \bar{\mu} + I\cdot\mu + \sigma_\xi \xi',$$

where $(\xi'_t)_{t\geq 0}$ are i.i.d. shocks with $|\xi'| \leq \bar{\xi} < \infty$ almost surely. Let $P(\cdot \mid \phi, I, \mu)$ denote the Markov kernel (c.d.f.) of ϕ'. Given $1 > \rho > 0$, the following hold:

(Supp) For each (ϕ, I, μ), the conditional support of ϕ' is the compact interval

$$\mathcal{Z}(\phi,I,\mu) := \left[\underline{\phi}'(\phi,I,\mu),\, \overline{\phi}'(\phi,I,\mu)\right] = \left[e^{\rho\ln\phi+\bar{\mu}+I\cdot\mu-\sigma_\xi\bar{\xi}},\, e^{\rho\ln\phi+\bar{\mu}+I\cdot\mu+\sigma_\xi\bar{\xi}}\right].$$

(MLR) Let $z(\phi,I,\mu) := \rho\ln\phi + \bar{\mu} + I\cdot\mu$. Then $X := \ln\phi' \mid (\phi,I,\mu)$ forms a location family on $[z(\phi,I,\mu)-\sigma_\xi\bar{\xi},\ z(\phi,I,\mu)+\sigma_\xi\bar{\xi}]$ that satisfies the monotone likelihood ratio (MLR) property in x as z increases. Since $x \mapsto e^x$ is strictly increasing, the MLR (and thus strict FOSD) holds for ϕ'.

(F1+) For any (ϕ,μ) with $\mu > 0$, $P(\cdot \mid \phi,1,\mu)$ strictly FOSD-dominates $P(\cdot \mid \phi,0,0)$.

(F2+) For fixed (I, μ), $P(\cdot \mid \phi, I, \mu)$ is strictly FOSD- and MLR-increasing in ϕ.

(Fμ+) For fixed ϕ and $I = 1$, $P(\cdot \mid \phi, 1, \mu)$ is strictly FOSD- and MLR-increasing in μ.

Lemma 8. Fix ϕ and define

$$\Delta(a, \phi) := V^1(a, \phi) - V^0(a, \phi),$$

where $V^I(a, \phi)$ is the value function conditional on the current-period discrete choice $I \in \{0, 1\}$. Let[47]

$$\mathcal{A}^1(\phi) := \{a \geq a_{\min} : \hat{y}(a, \phi) \geq \chi_f(\phi)\}.$$

For $a \notin \mathcal{A}^1(\phi)$, the choice $I = 1$ is infeasible in the current period and hence $V^1(a, \phi) = -\infty$ and $\Delta(a, \phi) = -\infty$.

Then, on $\mathcal{A}^1(\phi)$, $\Delta(a, \phi)$ is weakly increasing in a. Moreover, if U is strictly concave (e.g. CRRA with $\epsilon > 0$ or log utility) and $\chi_f(\phi) > 0$, then $\Delta(a, \phi)$ is strictly increasing in a on $\mathcal{A}^1(\phi)$.

Lemma 9. Fix $(\phi, I) \in [\underline{\phi}, \infty) \times \{0, 1\}$ and define the feasibility domain

$$\mathcal{A}^I(\phi) \;\equiv\; \big\{a \geq a_{\min} : \hat{y}(a, \phi) - I \cdot \chi_f(\phi) \geq 0\big\}.$$

For $a \in \mathcal{A}^I(\phi)$, define the conditional value

$$V^I(a, \phi) \;:=\; \max_{\substack{c \geq 0,\ a' \geq a_{\min},\ \mu \geq 0 \\ c + a' + \chi_v(\mu; \phi) + I \cdot \chi_f(\phi) \leq y(a, \phi)}} \big\{U(c) + \beta\, \mathbb{E}\big[V(a', \phi') \mid \phi, I, \mu\big]\big\}.$$

For $a \notin \mathcal{A}^I(\phi)$, set $V^I(a, \phi) := -\infty$. Then:

(i) For every $a \in \mathcal{A}^I(\phi)$, the feasible set

$$\Gamma^I(a, \phi) \;=\; \Big\{(a', \mu) \in [a_{\min}, +\infty) \times \mathbb{R}_+ :\ a' + \chi_v(\mu; \phi) \leq y(a, \phi) - I \cdot \chi_f(\phi)\Big\}$$

is nonempty, compact, and upper hemicontinuous in a.

(ii) The map $(a, a', \mu) \mapsto U\big(y(a, \phi) - a' - \chi_v(\mu; \phi) - I \cdot \chi_f(\phi)\big)$ is continuous. Moreover, for each fixed (ϕ, I),

$$(a', \mu) \mapsto \mathbb{E}\big[V(a', \phi') \mid \phi, I, \mu\big]$$

is continuous on $\mathbb{R}^2_+$.

[47] Note that $\hat{y}(a, \phi) = y(a, \phi) - a_{\min}$

(iii) For each $a \in \mathcal{A}^I(\phi)$, a maximizer exists and $V^I(\cdot, \phi)$ is continuous on $\mathcal{A}^I(\phi)$. Consequently, for each fixed ϕ, $\Delta(a, \phi) := V^1(a, \phi) - V^0(a, \phi)$ is continuous at all a where both regimes are feasible (i.e., $a \in \mathcal{A}^1(\phi)$), and $\Delta(a, \phi) = -\infty$ for $a \notin \mathcal{A}^1(\phi)$.

Lemma 10. Fix $\phi \geq \underline{\phi}$ and set $I = 1$. For $a \geq a_{\min}$ and $\mu \geq 0$, define

$$H(a, \mu; \phi) \ := \ \max_{a' \geq a_{\min}} \Big\{ U\Big(y(a, \phi) - a' - \chi_v(\mu; \phi) - \chi_f(\phi)\Big) \ + \ \beta\, \mathbb{E}\big[V(a', \phi') \mid \phi, 1, \mu\big] \Big\},$$

where feasibility is understood via the consumption constraint

$$c \ = \ y(a, \phi) - a' - \chi_v(\mu; \phi) - \chi_f(\phi) \ \geq \ 0,$$

and (only) infeasible choices with $c < 0$ are assigned value $-\infty$.

Then $H(\cdot, \cdot; \phi)$ has *increasing differences* in (a, μ) on the set where it is finite: for any $a_H > a_L$ and $\mu_H > \mu_L \geq 0$ such that all four values $H(a_i, \mu_j; \phi)$ are finite, we have

$$\big[H(a_H, \mu_H; \phi) - H(a_H, \mu_L; \phi)\big] \ \geq \ \big[H(a_L, \mu_H; \phi) - H(a_L, \mu_L; \phi)\big].$$

If, in addition, U is strictly concave on $(0, \infty)$ (e.g. CRRA with $\epsilon > 0$ or log utility), then $H(\cdot, \cdot; \phi)$ has strictly increasing differences on any region where the optimizer implies strictly positive consumption.

Lemma 11. Fix ϕ, and set $I = 1$. Define the μ-feasibility correspondence

$$\Gamma_\mu(a, \phi) \ := \ \Big\{\mu \geq 0 : \ \hat{y}(a, \phi) - \chi_f(\phi) - \chi_v(\mu; \phi) \geq 0\Big\},$$

where $\hat{y}(a, \phi) = y(a, \phi) - a_{\min}$. For each $a \geq a_{\min}$ such that $\Gamma_\mu(a, \phi) \neq \emptyset$, define the intensive-choice correspondence

$$\mathcal{M}(a, \phi) \ \equiv \ \arg \max_{\mu \in \Gamma_\mu(a, \phi)} H(a, \mu; \phi),$$

where $H(a, \mu; \phi)$ is defined in Lemma 10. Then:

(i) (*Existence and compactness*) For each a with $\Gamma_\mu(a, \phi) \neq \emptyset$, the set $\mathcal{M}(a, \phi)$ is nonempty and compact.

(ii) (*Monotonicity*) The correspondence $\mathcal{M}(\cdot, \phi)$ is nondecreasing in a in the strong set order. In particular, both the minimal and maximal selections from $\mathcal{M}(\cdot, \phi)$ are nondecreasing in a.

Lemma 12. Fix ϕ, and set $I = 1$. For each $a \geq a_{\min}$ with $\Gamma_\mu(a, \phi) \neq \emptyset$, let

$$\mathcal{M}(a, \phi) \ \equiv \ \arg \max_{\mu \in \Gamma_\mu(a,\phi)} H(a, \mu; \phi),$$

where $H(a, \mu; \phi)$ is defined in Lemma 10. Define the extremal selections

$$\underline{\mu}(a, \phi) := \min \mathcal{M}(a, \phi), \qquad \overline{\mu}(a, \phi) := \max \mathcal{M}(a, \phi).$$

Let $a_H > a_L$.

(i) **(Weak monotonicity; any $\epsilon \geq 0$).** Both $\underline{\mu}(\cdot, \phi)$ and $\overline{\mu}(\cdot, \phi)$ are nondecreasing in a. In particular, for any $\mu_L \in \mathcal{M}(a_L, \phi)$ there exists $\mu_H \in \mathcal{M}(a_H, \phi)$ such that $\mu_H \geq \mu_L$ (e.g. take $\mu_H = \overline{\mu}(a_H, \phi)$).

(ii) **(No downward shift when $\epsilon > 0$).** If $\epsilon > 0$ (CRRA strictly concave) and $\mu_L \in \mathcal{M}(a_L, \phi)$ with $\mu_L > 0$, then every $\mu_H \in \mathcal{M}(a_H, \phi)$ satisfies $\mu_H \geq \mu_L$.

(iii) **(Strict increase from an interior maximizer when $\epsilon > 0$).** Assume $\epsilon > 0$ and there exists a joint maximizer (a'_L, μ_L) of the *two-dimensional* conditional problem at (a_L, ϕ) such that $\mu_L > 0$ and the associated consumption is strictly positive,

$$c_L \ = \ y(a_L, \phi) - a'_L - \chi_v(\mu_L; \phi) - \chi_f(\phi) \ > \ 0.$$

Then every $\mu_H \in \mathcal{M}(a_H, \phi)$ satisfies $\mu_H > \mu_L$.

Proof of Proposition 2. Let $\mathcal{A}^1(\phi) = \{a \geq a_{\min} : \ \hat{y}(a, \phi) - \chi_f(\phi) \geq 0\}$ is the current-period feasibility set for undertaking R&D. We prove the extensive- and intensive-margin claims in turn.

Part (i): Extensive margin (threshold for I). Define the net gain from innovating:

$$\Delta(a, \phi) \ := \ V^1(a, \phi) - V^0(a, \phi).$$

By Lemma 9, $V^1(a, \phi) = -\infty$ for $a \notin \mathcal{A}^1(\phi)$, hence $\Delta(a, \phi) = -\infty < 0$ there and $I^*(a, \phi) = 0$.

Now restrict attention to $a \in \mathcal{A}^1(\phi)$. By Lemma 9 (iii), $\Delta(\cdot, \phi)$ is continuous on $\mathcal{A}^1(\phi)$, and by Lemma 8, $\Delta(\cdot, \phi)$ is weakly increasing in a on $\mathcal{A}^1(\phi)$ (strictly increasing if $\epsilon > 0$).

Define the cutoff

$$\underline{a}(\phi) \ := \ \inf\{a \in \mathcal{A}^1(\phi) : \ \Delta(a, \phi) \geq 0\},$$

with the convention $\inf \emptyset = \infty$. Monotonicity of $\Delta(\cdot, \phi)$ implies that, for $a \in \mathcal{A}^1(\phi)$,

$$\Delta(a, \phi) \geq 0 \quad \iff \quad a \geq \underline{a}(\phi).$$

Thus $I^*(a, \phi) = 1$ if and only if $a \in \mathcal{A}^1(\phi)$ and $a \geq \underline{a}(\phi)$.

Part (ii): Intensive margin (monotonicity and strict monotonicity of μ^*). Fix $\phi \geq \underline{\phi}$ and consider the conditional problem given $I = 1$. Recall the μ-feasible set

$$\Gamma_\mu(a, \phi) \;:=\; \big\{\mu \geq 0 : \; y(a, \phi) - \chi_f(\phi) - \chi_v(\mu; \phi) \geq 0\big\},$$

and the post-savings value

$$H(a, \mu; \phi) \;:=\; \max_{a' \geq a_{\min}} \Big\{U\big(y(a, \phi) - a' - \chi_v(\mu; \phi) - \chi_f(\phi)\big) + \beta\, \mathbb{E}[V(a', \phi') \mid \phi, 1, \mu]\Big\},$$

with infeasible choices (those implying $c < 0$ under the $U(0) = -\infty$ convention) assigned value $-\infty$ as in Lemma 10.

Step 1 (weak monotonicity). Lemma 10 shows that $H(\cdot, \cdot; \phi)$ has increasing differences in (a, μ) on its effective domain. Lemma 11 then implies that the optimizer correspondence $\mathcal{M}(\cdot, \phi)$ is nondecreasing in a in the strong set order. Since $\mathcal{M}(a, \phi)$ is nonempty and compact, the maximal selection $\mu^*(a, \phi) := \max \mathcal{M}(a, \phi)$ is well-defined and nondecreasing in a.

Step 2 (strict monotonicity when $\epsilon > 0$). Assume $\epsilon > 0$ and take any $a_H > a_L$ in the investment region (so the regime $I = 1$ is chosen and feasible at both wealth levels). Let (a'_L, μ_L) be a joint maximizer of the two-dimensional conditional problem at (a_L, ϕ); existence follows from Lemma 9 applied to $I = 1$.

We claim that this maximizer satisfies $\mu_L > 0$ and strictly positive consumption. Indeed, if $\mu_L = 0$, then the productivity transition under $(I, \mu) = (1, 0)$ coincides with that under $(I, \mu) = (0, 0)$ (Lemma 5), while the firm additionally pays the fixed cost $\chi_f(\phi) > 0$, so the regime $I = 1$ cannot be optimal—a contradiction. Hence $\mu_L > 0$. Moreover, under $\epsilon > 0$ with $\lim_{c \to 0^+} U(c) = +\infty$ (equivalently $c > 0$ in the admissible set), any optimizer must satisfy $c_L > 0$.

Therefore the hypotheses of Lemma 12 (iii) are met at (a_L, ϕ), which implies that for any $\mu_H \in \mathcal{M}(a_H, \phi)$, we have $\mu_H > \mu_L$. In particular, taking $\mu_H = \mu^*(a_H, \phi) = \max \mathcal{M}(a_H, \phi)$ and $\mu_L = \mu^*(a_L, \phi) = \max \mathcal{M}(a_L, \phi)$ yields

$$\mu^*(a_H, \phi) \;>\; \mu^*(a_L, \phi),$$

so $\mu^*(\cdot, \phi)$ is strictly increasing in a on the investment region.

□

C.4 Proof of Proposition 3

We begin by establishing the following lemmas, which facilitate the proof of Proposition 3 concerning the impact of η on R&D. The proofs of these lemmas, together with the similar proof for the impact of θ, are provided in the Online Appendix OA-D.5.1 and OA-D.5.2.

Lemma 13. The profit function $\pi(a, \phi; \theta, \eta)$ is non-decreasing in θ and η. Specifically:

(i) For unconstrained firms ($a \geq \hat{a}(\phi)$), the constraint is slack, so $\frac{\partial \pi}{\partial \theta} = \frac{\partial \pi}{\partial \eta} = 0$.

(ii) For constrained firms ($a < \hat{a}(\phi)$), profits are strictly increasing in financial development ($\frac{\partial \pi}{\partial \theta} > 0$ and $\frac{\partial \pi}{\partial \eta} > 0$) whenever the marginal product of capital exceeds the rental rate.

Consequently, the per-period resource function $y(a, \phi; \theta, \eta)$ is non-decreasing in θ and η. Furthermore, since the feasible set of the Bellman equation expands with θ and η while the objective function remains the same, the value function $V(a, \phi; \theta, \eta)$ is non-decreasing in θ and η (by the monotonicity of the Bellman operator).

Lemma 14. Fix the state (a, ϕ) and let $\tau \in \{\theta, \eta\}$ denote a financial development parameter. For clarity, write $y(\tau) := y(a, \phi; \tau)$.

The current-period utility from any given total of R&D expenditure plus savings

$$\mathcal{E} \;=\; \chi_v(\mu; \phi) + I \cdot \chi_f(\phi) + a',$$

defined by

$$\tilde{U}(\mathcal{E}, \tau) := U\big(y(\tau) - \mathcal{E}\big),$$

has *increasing differences* in $(\mathcal{E}, \tau)$.

More precisely, for any expenditure levels $\mathcal{E}' > \mathcal{E}$ and financial parameters $\tau_H > \tau_L$ such that $y(\tau_i) - \mathcal{E}' \geq 0$ for $i \in \{H, L\}$,

$$\tilde{U}(\mathcal{E}', \tau_H) - \tilde{U}(\mathcal{E}, \tau_H) \;\geq\; \tilde{U}(\mathcal{E}', \tau_L) - \tilde{U}(\mathcal{E}, \tau_L).$$

Lemma 18. Fix $\epsilon = 0$ and $\theta \geq 1$. Consider two intangible–collateral regimes: a benchmark regime $\eta_L = 0$ and a reform regime $\eta_H > 0$. For $j \in \{L, H\}$, let $V_j(a, \phi) := V(a, \phi; \theta, \eta_j)$, and define

$$\Delta V(a, \phi) \ := \ V_H(a, \phi) - V_L(a, \phi), \qquad (a, \phi) \in [a_{\min}, +\infty) \times [\underline{\phi}, +\infty).$$

For every a, and for any given current state (ϕ, I, μ) and bounded shock $|\xi'| \leq \bar{\xi}$, the one-step support

$$\mathcal{Z}(\phi, I, \mu) = \Big[e^{\rho \ln \phi + \bar{\mu} + I\mu - \sigma_\xi \bar{\xi}}, \ e^{\rho \ln \phi + \bar{\mu} + I\mu + \sigma_\xi \bar{\xi}}\Big]$$

is a compact interval, and $\Delta V(a, \cdot)$ is nonnegative and nondecreasing on $\mathcal{Z}(\phi, I, \mu)$.

Proof of the Impact of η on R&D. Throughout the proof we fix $\theta \geq 1$ and treat η as the only varying collateralizability parameter. For each η let $V(\cdot, \cdot; \eta)$ denote the unique value function in the weighted space $\mathcal{C}_\omega$ (See Lemma 4 in Online Appendix OA-D.4.1), and let $V^I(\cdot, \cdot; \eta)$, $I \in \{0, 1\}$, be the regime–specific value functions defined as in Lemma 9. We also write $y(a, \phi; \eta)$ for per-period resources to emphasize the dependence on η (holding θ fixed). Fix $0 = \eta_L < \eta_H$ and $\phi \geq \underline{\phi}$.

For each (a, ϕ, η) define the net value of innovation

$$\Delta(a, \phi; \eta) := V^1(a, \phi; \eta) - V^0(a, \phi; \eta).$$

By Lemma 8, for every (ϕ, η) the function $a \mapsto \Delta(a, \phi; \eta)$ is increasing on the region where $I = 1$ is feasible, and Proposition 2 implies that for each (ϕ, η) there exists a threshold $\underline{a}(\phi; \eta)$ such that $I^*(a, \phi; \eta) = 1$ if and only if $a \geq \underline{a}(\phi; \eta)$.

Step 1: Increasing differences of the flow objective in (I, η).

Fix a current state (a, ϕ) and a continuation choice vector $x = (a', \mu) \in [a_{\min}, +\infty) \times \mathbb{R}_+$. For $I \in \{0, 1\}$ and $\eta \in \{\eta_L, \eta_H\}$ define the one-period flow objective

$$\mathcal{O}(a, I; x, \phi; \eta) := U\big(y(a, \phi; \eta) - \mathcal{E}(I, \mu, a'; \phi)\big) + \beta \, \mathbb{E}\Big[V(a', \phi'; \eta) \mid \phi, I, \mu\Big],$$

where $\mathcal{E}(I, \mu, a'; \phi) := a' + \chi_v(\mu; \phi) + I \cdot \chi_f(\phi)$ is total expenditure on savings and R&D.

We claim that, for each fixed (a, ϕ, x), the map $(I, \eta) \mapsto \mathcal{O}(a, I; x, \phi; \eta)$ has (weakly) increasing differences on $\{0, 1\} \times \{\eta_L, \eta_H\}$.

(a) Current-period utility. Let $y(\eta) := y(a, \phi; \eta)$ and $x_0 := a' + \chi_v(\mu; \phi)$, $x_1 := x_0 + \chi_f(\phi)$. The current-period utility term equals $U(y(\eta) - x_1)$ under $I = 1$ and $U(y(\eta) - x_0)$ under $I = 0$. Lemma 14 (with $\tau \equiv \eta$) states that $(x, \eta) \mapsto U(y(\eta) - x)$ has increasing differences. Hence, for any $\eta_H > \eta_L$,

$$\big[U(y(\eta_H) - x_1) - U(y(\eta_H) - x_0)\big] - \big[U(y(\eta_L) - x_1) - U(y(\eta_L) - x_0)\big] \;\geq\; 0.$$

Thus the current-period utility component exhibits (weakly) increasing differences in (I, η).

(b) Continuation value. For fixed (a', ϕ, μ) define

$$\mathcal{V}_c^I(a', \phi; \mu, \eta) := \mathbb{E}\big[V(a', \phi'; \eta) \mid \phi, I, \mu\big], \qquad I \in \{0, 1\}.$$

For $\eta_H > \eta_L$ let

$$\Delta_{a', \eta_H, \eta_L}(\phi') := V(a', \phi'; \eta_H) - V(a', \phi'; \eta_L), \qquad \phi' \geq \underline{\phi}.$$

By Lemma 13, $V(a', \phi'; \eta)$ is nondecreasing in η, hence $\Delta_{a', \eta_H, \eta_L}(\phi') \geq 0$ for all ϕ'. Moreover, given $\epsilon = 0$, Lemma 18 implies that, for each fixed a', the function $\phi' \mapsto \Delta_{a', \eta_H, \eta_L}(\phi')$ is nonnegative and nondecreasing on $\mathcal{Z}(\phi, I, \mu)$.

Lemma 5, property (F1+), states that for any (ϕ, μ) with $\mu > 0$, the distribution of ϕ' under $(I = 1, \mu)$ strictly FOSD-dominates that under $(I = 0, 0)$; for $\mu = 0$ the two distributions coincide. Since $\Delta_{a', \eta_H, \eta_L}$ is nondecreasing, first-order stochastic dominance implies

$$\mathbb{E}\big[\Delta_{a', \eta_H, \eta_L}(\phi') \mid \phi, 1, \mu\big] \;\geq\; \mathbb{E}\big[\Delta_{a', \eta_H, \eta_L}(\phi') \mid \phi, 0, 0\big].$$

In terms of $\mathcal{V}_c^I$, this inequality is equivalent to

$$\big[\mathcal{V}_c^1(a', \phi; \mu, \eta_H) - \mathcal{V}_c^1(a', \phi; \mu, \eta_L)\big] - \big[\mathcal{V}_c^0(a', \phi; \mu, \eta_H) - \mathcal{V}_c^0(a', \phi; \mu, \eta_L)\big] \;\geq\; 0,$$

so the continuation-value component of $\mathcal{O}$ has (weakly) increasing differences in (I, η).

Combining (a) and (b), we conclude that, for every fixed (a, ϕ, x), the function

$$(I, \eta) \longmapsto \mathcal{O}(a, I; x, \phi; \eta)$$

has (weakly) increasing differences in (I, η) on $\{0, 1\} \times \{\eta_L, \eta_H\}$.

Step 2: From the flow objective to the net value of innovation.

To avoid dependence of the feasible set on (I, η), extend $\mathcal{O}$ to all $x = (a', \mu) \in [a_{\min}, +\infty) \times \mathbb{R}_+$ by setting

$$\mathcal{O}(a, I; x, \phi; \eta) := -\infty \qquad \text{whenever} \qquad y(a, \phi; \eta) - \mathcal{E}(I, \mu, a'; \phi) < 0.$$

Under this convention Lemma 9 implies that, for each (a, ϕ, I, η),

$$V^I(a, \phi; \eta) = \sup_{x \in [a_{\min}, +\infty) \times \mathbb{R}_+} \mathcal{O}(a, I; x, \phi; \eta),$$

and that the supremum is in fact attained.

Since $\mathcal{O}(a, I; x, \phi; \eta)$ has increasing differences in (I, η) pointwise in x and the feasible set does not depend on (I, η), Milgrom and Shannon's (**?**) maximization theorem implies that $(I, \eta) \mapsto V^I(a, \phi; \eta)$ inherits (weakly) increasing differences. Hence, for each fixed (a, ϕ) the function

$$\eta \longmapsto \Delta(a, \phi; \eta) = V^1(a, \phi; \eta) - V^0(a, \phi; \eta)$$

is nondecreasing on $[\eta_L, \eta_H]$.

Step 3: Extensive-margin comparative statics.

Fix $\phi \geq \underline{\phi}$. From Step 2 we have, for every $a \geq a_{\min}$,

$$\Delta(a, \phi; \eta_H) \;\geq\; \Delta(a, \phi; \eta_L).$$

Therefore

$$\big\{a \geq a_{\min} : \Delta(a, \phi; \eta_L) \geq 0\big\} \;\subseteq\; \big\{a \geq a_{\min} : \Delta(a, \phi; \eta_H) \geq 0\big\}.$$

By Lemma 8, for each (ϕ, η) the function $a \mapsto \Delta(a, \phi; \eta)$ is increasing on the set where $I = 1$ is feasible. Together with the threshold characterization in Proposition 2, this inclusion implies

$$\underline{a}(\phi; \eta_H) \;\leq\; \underline{a}(\phi; \eta_L) \qquad \text{for all } \phi \geq \underline{\phi},$$

which proves part (i).

Step 4: Intensive-margin comparative statics.

Now fix a state (a, ϕ) such that

$$I^*(a, \phi; \eta_L) = I^*(a, \phi; \eta_H) = 1.$$

For each $\eta \in \{\eta_L, \eta_H\}$ define the conditional value of choosing R&D intensity $\mu \geq 0$ (given $I = 1$) by

$$H(a, \mu; \phi, \eta) := \sup_{a' \geq a_{\min}} \Big\{ U\big(y(a, \phi; \eta) - a' - \chi_v(\mu; \phi) - \chi_f(\phi)\big) + \beta\, \mathbb{E}\big[V(a', \phi'; \eta) \mid \phi, 1, \mu\big] \Big\},$$

with infeasible choices (those implying $c < 0$) assigned value $-\infty$. By continuity of the objective and compactness of the feasible set in a' induced by the budget constraint, the supremum is attained. For each (a, ϕ, η) the argmax correspondence

$$\mathcal{M}(a, \phi; \eta) := \arg\max_{\mu \geq 0} H(a, \mu; \phi, \eta)$$

is therefore nonempty and compact.

We claim that, for each fixed (a, ϕ), the function $H(a, \mu; \phi, \eta)$ has (weakly) increasing differences in (μ, η) on $[0, \infty) \times \{\eta_L, \eta_H\}$.

(a) Current-period utility. Fix (a', ϕ) and define

$$\mathcal{E}(\mu) := a' + \chi_v(\mu; \phi) + \chi_f(\phi), \qquad y(\eta) := y(a, \phi; \eta).$$

Since $\chi_v(\mu; \phi)$ is strictly increasing in μ, $\mathcal{E}(\mu)$ is strictly increasing. Lemma 14 (with $\tau \equiv \eta$) implies that $(\mathcal{E}, \eta) \mapsto U(y(\eta) - \mathcal{E})$ has increasing differences. By the standard composition result for increasing differences (see e.g., **?**), it follows that

$$(\mu, \eta) \longmapsto U\big(y(\eta) - \mathcal{E}(\mu)\big)$$

has increasing differences in (μ, η).

(b) Continuation value. For fixed (a', ϕ) define

$$\mathcal{V}_c^1(a', \mu; \phi, \eta) := \mathbb{E}\big[V(a', \phi'; \eta) \mid \phi, 1, \mu\big].$$

Let $\Delta_{a', \eta_H, \eta_L}$ be as in Step 1(b). Then

$$\mathcal{V}_c^1(a', \mu; \phi, \eta_H) - \mathcal{V}_c^1(a', \mu; \phi, \eta_L) = \mathbb{E}\big[\Delta_{a', \eta_H, \eta_L}(\phi') \mid \phi, 1, \mu\big].$$

By Lemma 18, $\Delta_{a', \eta_H, \eta_L}$ is nonnegative and nondecreasing on $[\underline{\phi}, +\infty)$ under $\epsilon = 0$. Lemma 5, property (Fμ+), states that conditional on $(\phi, I = 1)$, the distribution of ϕ' is strictly FOSD- and MLR-increasing in μ. Hence, for any $\mu_H > \mu_L \geq 0$,

$$\mathbb{E}\big[\Delta_{a', \eta_H, \eta_L}(\phi') \mid \phi, 1, \mu_H\big] \;\geq\; \mathbb{E}\big[\Delta_{a', \eta_H, \eta_L}(\phi') \mid \phi, 1, \mu_L\big],$$

with strict inequality whenever the reform changes the continuation value with positive probability along the induced future states. Equivalently,

$$\left[\mathcal{V}_c^1(a', \mu_H; \phi, \eta_H) - \mathcal{V}_c^1(a', \mu_L; \phi, \eta_H)\right] - \left[\mathcal{V}_c^1(a', \mu_H; \phi, \eta_L) - \mathcal{V}_c^1(a', \mu_L; \phi, \eta_L)\right] \ \geq \ 0,$$

so the continuation-value component exhibits (weakly) increasing differences in (μ, η).

Combining (a) and (b), we conclude that $H(a, \mu; \phi, \eta)$ has (weakly) increasing differences in (μ, η) for each fixed (a, ϕ).

By Milgrom and Shannon's (**?**) monotone selection theorem, the argmax correspondence $\eta \mapsto \mathcal{M}(a, \phi; \eta)$ is nondecreasing in η in the strong set order:

$$\min \mathcal{M}(a, \phi; \eta_H) \ \geq \ \min \mathcal{M}(a, \phi; \eta_L), \qquad \max \mathcal{M}(a, \phi; \eta_H) \ \geq \ \max \mathcal{M}(a, \phi; \eta_L).$$

On the interior, where $\mathcal{M}(a, \phi; \eta)$ is single-valued, this implies

$$\mu^*(a, \phi; \eta_H) \ \geq \ \mu^*(a, \phi; \eta_L)$$

whenever $I^*(a, \phi; \eta_L) = I^*(a, \phi; \eta_H) = 1$. This proves part (ii) and completes the proof of the proposition. □

D Role of Elasticity of Intertemporal Substitution

Figures A.1 and A.2 illustrate how the elasticity of intertemporal substitution shapes the policy functions for optimal R&D and savings, respectively, under various parameters of the IP-backed lending policy. The key result is that the effect of IP-backed lending on R&D weakens as ϵ increases, driven by a more significant wealth effect and an increased utility cost of R&D.

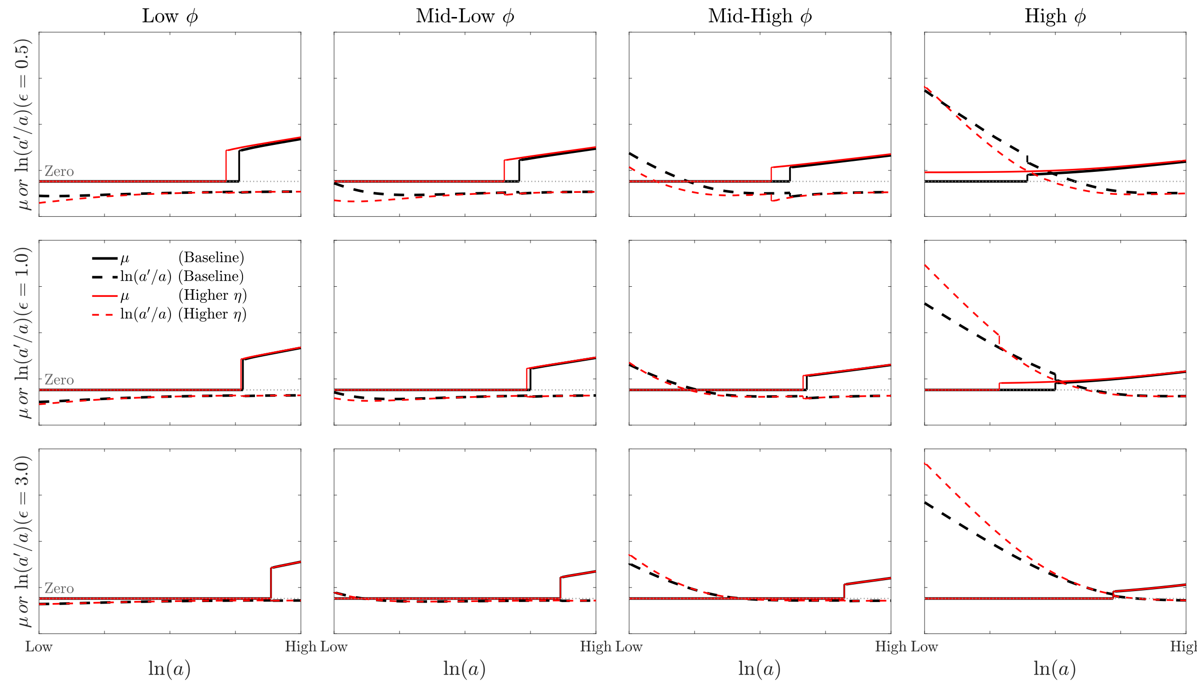


Figure A.1: Optimal R&D and Saving as Functions of Net Worth across Different Productivity Levels: The Role of Elasticitys of Intertemporal Substitution (Baseline VS IP-backed Lending Policy)

Note: the scales of y-axis for μ and $\ln(a'/a)$ are the same.

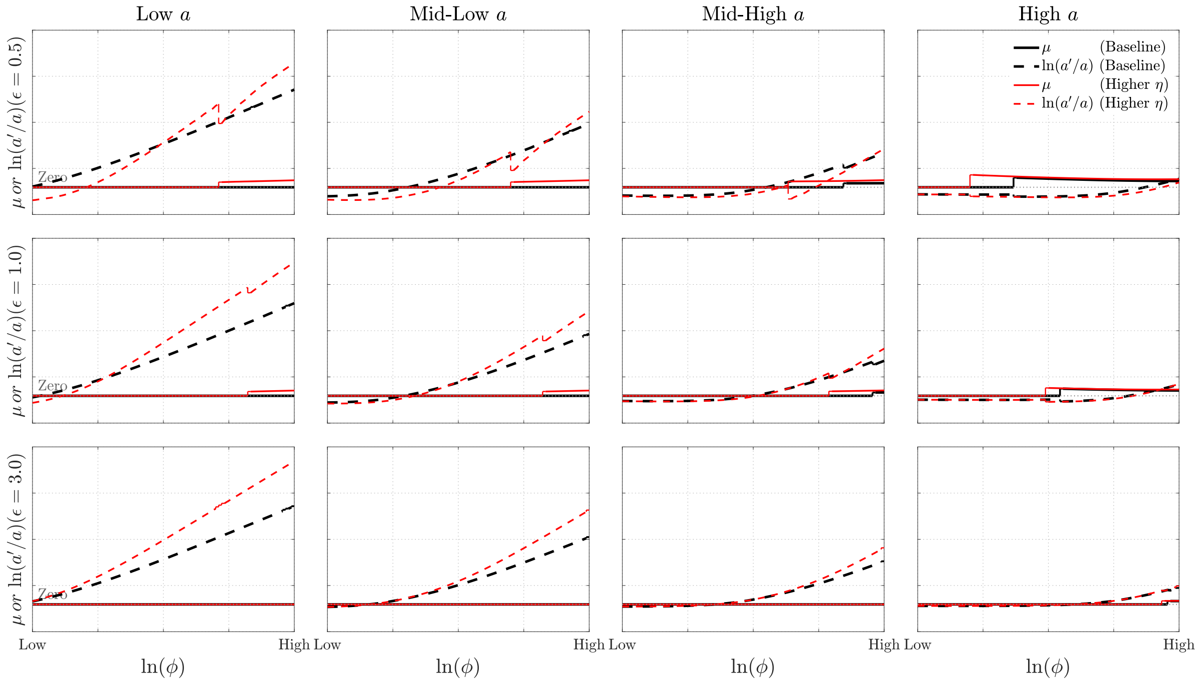


Figure A.2: Optimal R&D and Saving as Functions of Productivity across Different Net Worth Levels: The Role of Elasticitys of Intertemporal Substitution (Baseline VS IP-backed Lending Policy)

Note: the scales of y-axis for μ and $\ln(a'/a)$ are the same.

E Aggregation and TFP Decomposition

The total capital and labor input used in the economy are given by:

$$
\begin{aligned}
K_t =& N \cdot \Gamma(w) \int M(a, \phi; w)^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} dG_t(a, \phi), \\
L_t =& N \cdot \frac{\alpha_l}{\alpha_k w} \Gamma(w) \int M(a, \phi; w)^{-\frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} dG_t(a, \phi),
\end{aligned}
$$

where $G_t(a, \phi)$ is the distribution of firms over states at time t. The aggregate output is given by:

$$
Q_t = \Gamma(w)^{\alpha_k+\alpha_l} \left(\frac{\alpha_l}{\alpha_k w}\right)^{\alpha_l} \left[\int \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} M(a, \phi)^{\frac{-\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} dG_t(a, \phi)\right]^{\frac{\sigma}{\sigma-1}}
$$

Then the actual TFP (ATFP) is given by:

$$
\mathrm{ATFP}_t = N^{\frac{1}{\sigma-1}} \frac{\left[\int \phi^{\sigma\tilde{\alpha}_\phi} M(a, \phi)^{-\sigma\tilde{\alpha}_k} dG_t(a, \phi)\right]^{\frac{\sigma}{\sigma-1}-\alpha_l}}{\left[\int \phi^{\sigma\tilde{\alpha}_\phi} M(a, \phi)^{-\sigma(1-\tilde{\alpha}_l)} dG_t(a, \phi)\right]^{\alpha_k}}
$$

In the absence of borrowing constraints, i.e., $M(a, \phi) = r_t + \delta$, the efficient TFP (ETFP) is given by:

$$\text{ETFP}_t = N^{\frac{1}{\sigma-1}} \left(\int \phi^{(\sigma-1)\alpha_\phi} dG_t(a,\, \phi) \right)^{\frac{1}{\sigma-1}}$$

The ETFP measures the aggregate productivity when the capital is allocated efficiently across firms given the fundamental productivity distribution. However, due to financial frictions, the actual allocation of capital deviates from the efficient one, leading to a gap between ATFP and ETFP.

F Local Identification

Table A.2 displays the elasticity of each moment with respect to each parameter $\left(\frac{\partial \log(\text{Moment}_i)}{\partial \log(\text{Parameter}_j)}\right)$, calculated at the estimated values. Although the numerical magnitudes could be sensitive to the step size of the parameter changes, the signs of the elasticities are robust, indicating the qualitative link between moments and parameters.

Table A.2: Local Identification

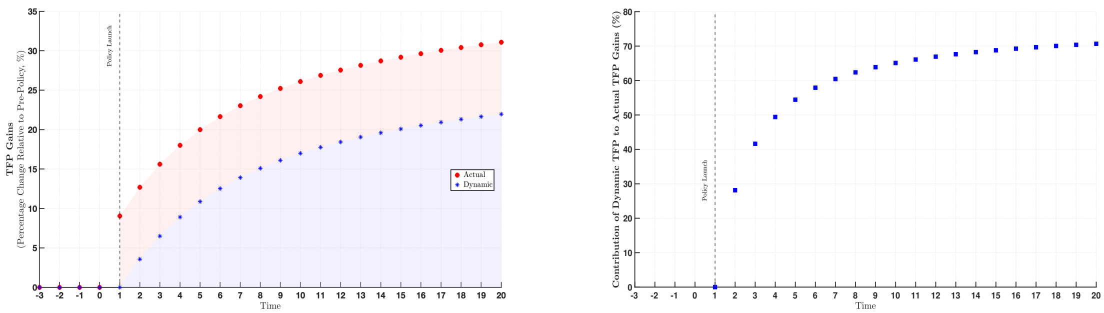
	β	ρ_ϕ	σ_ξ	f	γ	ν	ζ_1	ζ_2	θ	η
IQR of profitability	1.948	227.04	0.806	-0.048	-0.003	-0.090	-0.003	-0.094	0.030	0.031
IDR of profitability	1.530	126.29	0.773	-0.049	-0.004	-0.102	-0.004	-0.124	0.033	0.035
Median leverage ratio (k/net worth)	-7.554	6.708	-2.505	-0.033	-0.002	-0.057	-0.001	-0.047	0.028	0.169
Share of firms in R&D	30.934	68.100	-2.781	-1.042	-0.054	-1.234	-0.058	-1.161	0.452	0.488
$\tilde{\rho}_{\text{tfp}}$	-0.132	2.200	-0.000	0.001	0.000	0.002	-0.000	-0.010	-0.001	-0.000
$\tilde{\mu}_{\text{tfp}}$	4.995	-6.309	-0.003	0.115	-0.010	-0.653	0.021	-0.742	0.108	0.075
$\tilde{\beta}^{\text{extensive}}_{\text{tfp}}$	5.343	-11.464	-2.828	-0.563	-0.041	-0.710	-0.088	-1.380	0.385	0.506
$\tilde{\beta}^{\text{extensive}}_{\text{net worth}}$	11.549	-6.057	-2.194	-0.781	-0.032	-0.896	-0.010	-0.431	0.169	-0.012
$\tilde{\beta}^{\text{intensive}}_{\text{tfp}}$	-0.916	10.373	-0.339	0.057	0.005	-0.030	0.014	-1.455	0.144	0.178
$\tilde{\beta}^{\text{intensive}}_{\text{net worth}}$	0.150	-2.556	0.161	-0.084	0.002	-0.169	-0.016	-0.600	-0.102	-0.188

G Partial Equilibrium Results

In this section, we examine the partial equilibrium effects of the IP-backed lending policy on aggregate TFP and decompose these effects into allocation efficiency improvements and technology improvements. In this analysis, we hold wages fixed at their pre-policy level and calculate the transition paths of actual TFP and effective TFP from the initial steady state to the new steady state following the policy implementation.

Figure A.3a presents the evolution of TFP gains over time. The red line (circles) demonstrates that actual aggregate TFP experiences an immediate jump at the policy's inception, followed by gradual increases that converge to approximately 35% above the pre-policy level. In other words, absent wage adjustments, the long-run increase in aggregate TFP reaches 35%. The blue line (stars) in the same figure shows that dynamic aggregate TFP gains exhibit no initial discontinuous jump but instead increase gradually, converging to approximately 25%. The gap between these two lines represents static gains from allocation efficiency, which amounts to roughly 10% in the long run.

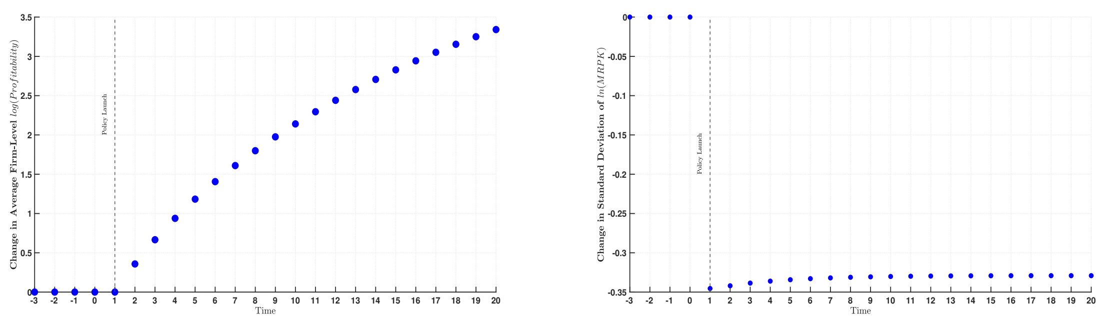
The initial gains in actual aggregate TFP primarily stem from allocation efficiency improvements, as the effects of relaxed financial constraints on capital investment materialize immediately. In contrast, medium-run and long-run gains in actual aggregate TFP are predominantly driven by technological advancement, since the impact of R&D investment on technology requires time to manifest. Figure A.3b illustrates the contribution of technology improvement to actual aggregate TFP gains, which increases from 0% initially to 70% in the long run.



(a) Actual TFP Gains and Dynamic TFP Gains (b) Contribution of Dynamic TFP Gains

Figure A.3: Effects in Partial Equilibrium

Note: The left panel displays the percentage change for each variable relative to its corresponding value prior to the policy implementation. The right panel illustrates the share of Dynamic TFP gains within Actual TFP gains.



(a) RTFP (b) Standard Deviation of ln(MRPK)

Figure A.4: Effects in Partial Equilibrium on RTFP and Std of ln(MRPK)

Note: The left panel displays the percentage change in average firm-level RTFP relative to pre-policy levels. The right panel illustrates the level change in the standard deviation of $ln(MRPK)$ relative to pre-policy levels.

H General Equilibrium Results

H.1 Transitional Dynamics of the Equilibrium Wage

From the partial equilibrium analysis, technology improvement appears to contribute most significantly to aggregate TFP in the long run. However, if the policy was implemented economy-wide, production costs (particularly wages) will increase, thereby reducing the benefits of investment in both R&D and capital.

Figure A.5 displays the transition dynamics of wages. As shown, wages initially increase dramatically due to improved capital allocation efficiency, which drives up labor demand. Subsequently, wages continue to increase gradually. In the long run, wages rise by 16% compared to their pre-policy level. Since we do not explicitly model worker utility, this figure can serve as a rough approximation of welfare gains for workers.

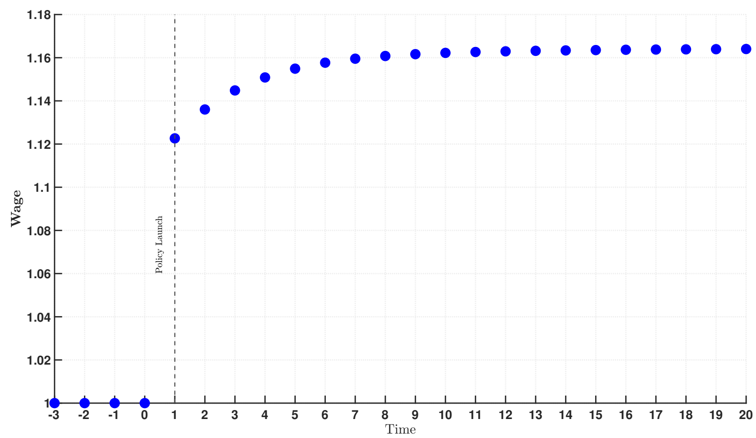


Figure A.5: Wage Path

Note: This figure illustrates the level change in the equilibrium wage, with the pre-policy wage normalized to 1.

H.2 Policy Impact on Other Economic Indicators in General Equilibrium

The following figures summarize the general equilibrium effects of the IP-backed lending policy. Figure A.6 illustrates the contribution of dynamic TFP gains to actual TFP gains. Figures A.7a and A.7b display the policy's impact on the extensive and intensive margin of R&D expenditure.

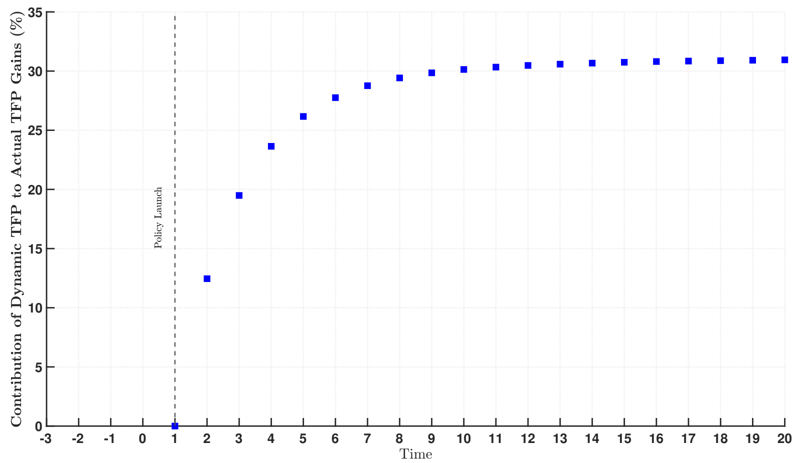
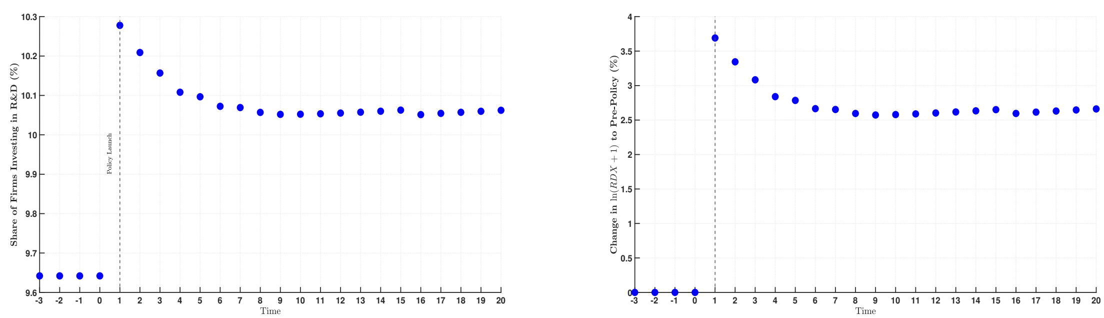


Figure A.6: Contribution of Dynamic TFP Gains from IP-Backed Lending in General Equilibrium

Note: the figure illustrates the share of Dynamic TFP gains within Actual TFP gains.



(a) Average $\mathbb{1}\{\text{R\&D} > 0\}$

(b) Average $\ln(\text{RDX} + 1)$

Figure A.7: Policy Impact on Extensive and Intensive Margin of R&D Expenditure

Note: Panel a displays the fraction of firms investing in R&D. Panels b plot deviations from pre-policy levels for the average $\ln(RDX + 1)$.

Online Appendix

for "Intellectual Property Financing, Innovation, and Aggregate TFP"

Zhiyuan Chen[†] Guanliang Hu[‡]

[†] Business School, Renmin University

[‡] Department of Economics and Finance, City University of Hong Kong

OA-A Data Appendix

OA-A.1 Summary Statistics

Table OA-1 and OA-2 displays summary statistics for the key variables, covering both the pre-policy period (2005–2007) and the post-policy period (2012–2014).

Table OA-1: Summary Statistics of Key Variables (2005–2007)

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Variable	ln(VA)	ln(K)	ln(L)	ln(RTFP)	ln(NW)	$\mathbb{I}_{\text{RDX}>0}$	$\ln(\text{RDX})_{>0}$	$\ln(\text{RDX}+1)$
Mean	8.693	8.414	4.655	5.626	8.763	0.103	5.479	0.567
Std. dev.	1.292	1.645	1.080	1.007	1.584	0.304	2.212	1.812
10%	7.241	6.444	3.401	4.458	6.845	0.000	2.708	0.000
25%	7.805	7.372	3.912	4.946	7.707	0.000	3.912	0.000
50%	8.539	8.350	4.564	5.537	8.669	0.000	5.398	0.000
75%	9.433	9.403	5.298	6.224	9.711	0.000	6.962	0.000
90%	10.377	10.475	6.040	6.941	10.815	1.000	8.359	1.609

Note: This table presents summary statistics for the primary variables used in the analysis. The number of observations is 661,567. VA = Value Added; K = Capital Stock; L = Labor (number of employees); RTFP = Revenuse-based productivity; NW = Net Worth; $\mathbb{I}_{\text{RDX}>0}$ is an indicator variable equal to 1 if R&D expenditures are positive and 0 otherwise; $\ln(\text{RDX})_{>0}$ represents the natural log of R&D expenditures for the subsample of firms with non-zero R&D; and $\ln(1+\text{RDX})$ is the log-transformed R&D expenditure for the full sample.

Table OA-2: Summary Statistics of Key Variables (2012–2014)

Variable	(1) ln(VA)	(2) ln(K)	(3) ln(L)	(4) ln(RTFP)	(5) ln(NW)	(6) $\mathbb{I}_{\text{RDX}>0}$	(7) $\ln(\text{RDX})_{>0}$	(8) $\ln(\text{RDX}+1)$
Mean	9.179	8.747	4.546	6.161	9.780	0.120	7.760	0.928
Std. dev.	1.744	2.228	1.375	1.197	1.965	0.324	2.083	2.619
10%	6.996	5.900	2.773	4.757	7.258	0.000	4.927	0.000
25%	8.101	7.380	3.689	5.378	8.530	0.000	6.799	0.000
50%	9.148	8.878	4.549	6.079	9.782	0.000	8.097	0.000
75%	10.268	10.198	5.429	6.900	11.071	0.000	9.095	0.000
90%	11.395	11.424	6.268	7.729	12.242	1.000	9.979	5.938

Note: This table presents summary statistics for the primary variables used in the analysis. The number of observations is 437,002. VA = Value Added; K = Capital Stock; L = Labor (number of employees); RTFP = Revenuse-based productivity; NW = Net Worth; $\mathbb{I}_{\text{RDX}>0}$ is an indicator variable equal to 1 if R&D expenditures are positive and 0 otherwise; $\ln(\text{RDX})_{>0}$ represents the natural log of R&D expenditures for the subsample of firms with non-zero R&D; and $\ln(1+\text{RDX})$ is the log-transformed R&D expenditure for the full sample.

OA-A.2 Pilot Cities

The pilot cities and their designation years are shown in Table OA-3 and Figure OA-1.

Designation Year	**Pilot Cities**
2009	Changchun, Wuxi, Wenzhou, Nanchang, Yichang, Xiangtan, Guangzhou, Foshan, Dongguan, Chengdu, Yinchuan, Shizuishan, Wuzhong, Zhongwei
2010	Tianjin, Shanghai, Zhenjiang, Wuhan
2011	Chongqing
2012	Bengbu, Quanzhou, Zhangzhou, Fuzhou, Weifang, Weihai, Mianyang
2013	Binzhou

Table OA-3: List of Pilot Cities

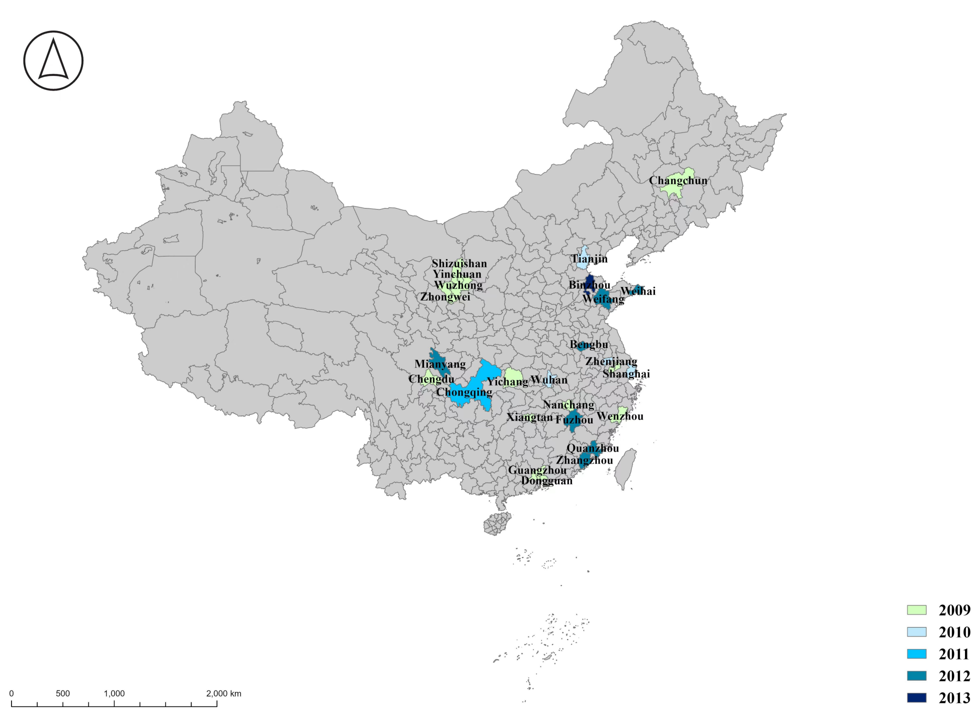


Figure OA-1: Pilot Cities Map

OA-B Micro-foundation of IP-Backed Lending

We provide a theoretical explanation for how IP-backed lending policies shape banks' willingness to extend loans secured by IP. The model in this section illustrates how adverse selection, arising from asymmetric information about IP quality, can cause credit markets to freeze. We show that government interventions—including IP assessment facilities, subsidies, and quality improvements—are essential to alleviate this friction and increasing lending.

Specifically, we prove four key results: (1) Absent IP evaluation, no loans are issued under plausible conditions; (2) Establishing assessment facilities raises banks' willingness to lend; (3) Providing subsidies further incentivizes lending; (4) Encouraging high-quality IP development boosts loan issuance. These results provide a rationale for policies that enhance the pledgeability of intangible assets.

Please note that our structural model in the main text does not incorporate the bank decision-making process described in this section. The purpose of this section is to demonstrate that each policy tool can enhance banks' incentives to increase lending to firms. Through this theoretical framework, we provide justification for why the pledgeability of intangibles increases following the policy implementation.

OA-B.1 Environment Setup

We consider a static environment with risk-neutral firms and risk-neutral banks. The risk-free interest rate is normalized to zero, implying that banks require a net expected return of zero on loans.

The Firm. Each firm possesses a project requiring an upfront investment cost $c > 0$. The firm has no internal funds and must borrow to finance the project. The project's IP quality is binary: $\mu_q \in \{\mu_q^L, \mu_q^H\}$, with $\mu_q^L < \mu_q^H$. The prior probability of high quality is $\mathbb{P}(\mu_q = \mu_q^H) = P \in (0, 1)$.

Upon financing, the project yields output $q \sim \mathbb{N}(\mu_q, \sigma_q^2)$. We analyze the limiting case $\sigma_q \to 0$, where output converges to its mean deterministically.

If financed, the firm borrows c at a gross repayment $D = (1 + r) \cdot c$. Under limited liability, if $q \geq D$, the firm repays D and earns $q - D$. If $q < D$, it defaults and earns zero. In the limit $\sigma_q \to 0$, high-quality firms ($\mu_q^H > D$) always repay and earn positive profit, while low-quality firms always default. Firms have incentives to borrow as long as expected profits exceed zero, which holds absent significant default costs or self-financing options.[48]

The Bank. Banks cannot directly observe μ_q but receive a noisy signal $s = \mu_q + e$, where $e \sim \mathbb{N}(0, \sigma_e^2)$. Upon default, the bank recovers a fraction $\zeta \in (0, 1)$ of the output as liquidation value. Banks lend if the expected return meets or exceeds their opportunity cost (normalized to zero).

Assumptions. To generate adverse selection and potential market failure within the model, we make the following assumptions:

Assumption 3. $\mu_q^L < (1 + r) \cdot c < \mu_q^H$.

[48] If self-financing were possible, there would exist a threshold interest rate $\hat{r}(\mu_q)$ above which firms opt out; we abstract from this for simplicity.

This ensures that, as $\sigma_q \to 0$, high-quality projects succeed with certainty, while low-quality ones fail. Additionally, we impose:

Assumption 4. $P \cdot (1 + r) \cdot c + (1 - P) \cdot \zeta \cdot \mu_q^L < c$.

This condition holds when P is low, c is high, ζ is low, or μ_q^L is low, leading to credit rationing under asymmetric information.

OA-B.2 Pre-Policy Analysis

The bank's posterior belief of high quality, $\pi(s) \equiv \mathbb{P}(\mu_q = \mu_q^H \mid s)$, is given by Bayes' rule:

$$\pi(s) = \frac{P \cdot \phi\left(\frac{s-\mu_q^H}{\sigma_e}\right)}{P \cdot \phi\left(\frac{s-\mu_q^H}{\sigma_e}\right) + (1 - P) \cdot \phi\left(\frac{s-\mu_q^L}{\sigma_e}\right)}, \tag{OA-1}$$

where $\phi(\cdot)$ is the standard normal PDF. Since the normal distribution satisfies the Monotone Likelihood Ratio Property (MLRP), the posterior probability $\pi(s)$ is strictly increasing in s.

In the limit $\sigma_q \to 0$, the bank's expected profit from lending is:

$$V(s) = \pi(s)\,(r \cdot c) + (1 - \pi(s))\left(\zeta \cdot \mu_q^L - c\right). \tag{OA-2}$$

The bank lends if and only if $V(s) \geq 0$.

OA-B.2.1 Market Freeze without Information

Proposition 4. If $\sigma_e \to \infty$ and Assumption 4 holds, no loans are issued.

Proof. As $\sigma_e \to \infty$, the signal becomes uninformative, so $\pi(s) \to P$ for all s. Thus, the expected value of lending converges to the unconditional mean: $V = P \cdot (r \cdot c) + (1 - P) \cdot (\zeta \cdot \mu_q^L - c)$. By Assumption 4, this value is strictly negative. Consequently, banks reject all loan applications. □

Remark 1. This benchmark captures pre-policy environments where banks lack expertise to evaluate IP, leading to adverse selection and credit market failure.

OA-B.2.2 Lending with Noisy Information

Lemma 1. Under Assumptions 3–4 and finite σ_e, there exists a unique threshold $\underline{s}$ such that banks lend if and only if $s \geq \underline{s}$, where:

$$\underline{s} = \frac{\mu_q^H + \mu_q^L}{2} + \frac{\sigma_e^2}{\mu_q^H - \mu_q^L} \ln \left(\frac{(1-P) \cdot (c - \zeta \cdot \mu_q^L)}{P \cdot r \cdot c} \right). \tag{OA-3}$$

Proof. Set $V(s) = 0$ in (OA-2):

$$\pi(s) \cdot r \cdot c = (1 - \pi(s)) \cdot (c - \zeta \cdot \mu_q^L) \implies \frac{\pi(s)}{1 - \pi(s)} = \frac{c - \zeta \cdot \mu_q^L}{r \cdot c}.$$

The posterior odds ratio for normal distributions is given by:

$$\frac{\pi(s)}{1 - \pi(s)} = \frac{P}{1-P} \exp \left(\frac{2s \cdot (\mu_q^H - \mu_q^L) - [(\mu_q^H)^2 - (\mu_q^L)^2]}{2\sigma_e^2} \right).$$

Equating the odds and taking natural logarithms yields the expression in (OA-3). Since $\pi(s)$ is strictly increasing in s (due to MLRP), $V(s)$ is strictly increasing, ensuring the threshold $\underline{s}$ is unique. $\square$

Corollary 1. The probability of approval is strictly higher for high-quality firms: $\mathbb{P}(s > \underline{s} \mid \mu_q^H) > \mathbb{P}(s > \underline{s} \mid \mu_q^L)$.

OA-B.3 The Effects of Policy Tools

We examine the effect of different policy tools on the threshold $\underline{s}$, where lower values correspond to increased lending.

OA-B.3.1 Enhancing Precision in IP Assessment

Proposition 5. Lower σ_e (indicating higher assessment precision) reduces the lending threshold $\underline{s}$, expanding credit access by mitigating adverse selection.

Proof. Consider the threshold equation (OA-3). Let $K = \frac{(1-P)\cdot(c-\zeta\cdot\mu_q^L)}{P\cdot r\cdot c}$. By Assumption 4, the unconditional expected profit is negative, which implies $P \cdot r \cdot c < (1-P) \cdot (c - \zeta \cdot \mu_q^L)$. Therefore, $K > 1$ and $\ln(K) > 0$.

Since $\ln(K) > 0$ and $\mu_q^H > \mu_q^L$, the term multiplying σ_e^2 is positive. Thus, $\underline{s}$ is strictly increasing in σ_e^2. A reduction in σ_e lowers $\underline{s}$, increasing the probability that a firm qualifies for a loan. □

OA-B.3.2 Lending Subsidies

Suppose the government subsidizes loans at rate $\tau \in [0, 1)$, reducing the bank's effective cost to $(1 - \tau)c$.

Proposition 6. The threshold $\underline{s}(\tau)$ is strictly decreasing in τ, for $\tau < 1 - \frac{(1+r)\cdot\zeta\cdot\mu_q^L}{c}$.

Proof. The modified profit is $V(s; \tau) = \pi(s) \cdot [(r + \tau)c] + (1 - \pi(s)) \cdot [\zeta \cdot \mu_q^L - (1 - \tau)c]$. Setting $V = 0$ yields:

$$\underline{s}(\tau) = \frac{\mu_q^H + \mu_q^L}{2} + \frac{\sigma_e^2}{\mu_q^H - \mu_q^L} \ln \left(\frac{1 - P}{P} \cdot \frac{(1 - \tau)c - \zeta\mu_q^L}{(r + \tau)c} \right).$$

Let $\Lambda(\tau) = \frac{(1-\tau)c-\zeta\cdot\mu_q^L}{(r+\tau)c}$. Then:

$$\frac{\partial\Lambda}{\partial\tau} = \frac{-c[(r + \tau)c + (1 - \tau)c - \zeta \cdot \mu_q^L]}{[(r + \tau)c]^2} < 0.$$

Since $\Lambda(\tau)$ decreases in τ, $\ln(\Lambda(\tau))$ decreases, causing $\underline{s}(\tau)$ to decrease. For $\tau \geq 1 - \frac{(1+r)\zeta\cdot\mu_q^L}{c}$, banks lend to all firms regardless of the signal. □

Remark 2. Subsidies effectively raise returns on high-quality loans and reduce losses on low-quality ones, encouraging broader lending.

OA-B.3.3 Quality Improvement

Policies that increase the prior P (e.g., by promoting high-quality innovation) improve the applicant pool.

Proposition 7. The threshold $\underline{s}(P)$ is strictly decreasing in P.

Proof. From (OA-3), $\underline{s}(P)$ depends on P through the term $\ln\left(\frac{1-P}{P} \cdot C\right)$, where $C > 0$ is a constant independent of P. Since the odds ratio $\frac{1-P}{P}$ is strictly decreasing in P, the logarithm is also decreasing. Thus, $\underline{s}(P)$ decreases as P increases. □

Remark 3. Higher P reduces adverse selection by improving average quality, lowering the signal strength required for approval.

OA-C Empirical Analysis

OA-C.1 Robustness Check — Using Alternative Fixed Effect Controls

We conduct robustness checks on Table 1 using alternative fixed effect controls. We report results for models: (1) without fixed effects, (2) with city fixed effects, (3) with year fixed effects, (4) with city and year fixed effects, (5) with city, year, and industry fixed effects, and (6) with city and industry-year fixed effects. The results for the corresponding dependent variables are reported in Tables OA-4–OA-7.

Table OA-4: The Effect of Policy on the Decision to Undertake R&D Investment

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Dependent Variable	$\mathbb{I}_{\text{RDX}>0}$	$\mathbb{I}_{\text{RDX}>0}$	$\mathbb{I}_{\text{RDX}>0}$	$\mathbb{I}_{\text{RDX}>0}$	$\mathbb{I}_{\text{RDX}>0}$	$\mathbb{I}_{\text{RDX}>0}$
Pilot · After	0.0367***	0.0314***	0.0363***	0.0310***	0.0263***	0.0245***
	(0.0014)	(0.0014)	(0.0014)	(0.0014)	(0.0014)	(0.0014)
Observations	1,258,798	1,258,798	1,258,798	1,258,798	1,258,798	1,258,798
R-squared	0.002	0.018	0.002	0.019	0.073	0.080
City FE	No	Yes	No	Yes	Yes	Yes
Year FE	No	No	Yes	Yes	Yes	No
Industry FE	No	No	No	No	Yes	No
Industry-Year FE	No	No	No	No	No	Yes

Note: Robust standard deviations are provided in parentheses.

*** p<0.01, ** p<0.05, * p<0.1

Table OA-5: The Effect of Policy on R&D Expenditure

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Dependent Variable	ln(RDX+1)	ln(RDX+1)	ln(RDX+1)	ln(RDX+1)	ln(RDX+1)	ln(RDX+1)
Pilot · After	0.3089***	0.2709***	0.3048***	0.2666***	0.2183***	0.1962***
	(0.0104)	(0.0107)	(0.0104)	(0.0108)	(0.0106)	(0.0106)
Observations	1,258,720	1,258,720	1,258,720	1,258,720	1,258,707	1,258,633
R-squared	0.005	0.021	0.005	0.021	0.083	0.092
City FE	No	Yes	No	Yes	Yes	Yes
Year FE	No	No	Yes	Yes	Yes	No
Industry FE	No	No	No	No	Yes	No
Industry-Year FE	No	No	No	No	No	Yes

Note: Robust standard deviations are provided in parentheses.

*** p<0.01, ** p<0.05, * p<0.1

Table OA-6: The Effect of Policy on Revenue-Based Firm-Level TFP

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Dependent Variable	ln(RTFP)	ln(RTFP)	ln(RTFP)	ln(RTFP)	ln(RTFP)	ln(RTFP)
Pilot · After	0.3440***	0.3379***	0.3355***	0.3281***	0.2993***	0.2893***
	(0.0055)	(0.0055)	(0.0054)	(0.0055)	(0.0047)	(0.0047)
Observations	753,097	753,097	753,097	753,097	753,097	753,097
R-squared	0.087	0.129	0.107	0.149	0.366	0.376
City FE	No	Yes	No	Yes	Yes	Yes
Year FE	No	No	Yes	Yes	Yes	No
Industry FE	No	No	No	No	Yes	No
Industry-Year FE	No	No	No	No	No	Yes

Note: Robust standard deviations are provided in parentheses.

*** p<0.01, ** p<0.05, * p<0.1

Table OA-7: The Effect of Policy on the Standard Deviation of Log MRPK

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Dependent Variable	$\sigma^{\text{mrpk}}_{c,s,t}$	$\sigma^{\text{mrpk}}_{c,s,t}$	$\sigma^{\text{mrpk}}_{c,s,t}$	$\sigma^{\text{mrpk}}_{c,s,t}$	$\sigma^{\text{mrpk}}_{c,s,t}$	$\sigma^{\text{mrpk}}_{c,s,t}$
Pilot · After	-0.1321***	-0.1174***	-0.1322***	-0.1177***	-0.1067***	-0.1000***
	(0.0102)	(0.0101)	(0.0102)	(0.0101)	(0.0101)	(0.0101)
Observations	106,830	106,830	106,830	106,830	106,830	106,830
R-squared	0.114	0.139	0.114	0.139	0.163	0.184
City FE	No	Yes	No	Yes	Yes	Yes
Year FE	No	No	Yes	Yes	Yes	No
Industry FE	No	No	No	No	Yes	No
Industry-Year FE	No	No	No	No	No	Yes

Note: Robust standard deviations are provided in parentheses.

*** p<0.01, ** p<0.05, * p<0.1

OA-C.2 Percentage Changes in Dispersion of ln(MRPK) Post-Policy

Table OA-8 reports the effect of the policy on the dispersion of ln(MRPK), expressed as percentage changes.

Table OA-8: The Effect of Policy on the Log Standard Deviation of Log MRPK

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Dependent Variable	$\ln(\sigma^{\text{mrpk}}_{c,s,t})$	$\ln(\sigma^{\text{mrpk}}_{c,s,t})$	$\ln(\sigma^{\text{mrpk}}_{c,s,t})$	$\ln(\sigma^{\text{mrpk}}_{c,s,t})$	$\ln(\sigma^{\text{mrpk}}_{c,s,t})$	$\ln(\sigma^{\text{mrpk}}_{c,s,t})$
Pilot · After	-0.1601***	-0.1407***	-0.1599***	-0.1405***	-0.1363***	-0.1300***
	(0.0123)	(0.0123)	(0.0123)	(0.0123)	(0.0123)	(0.0123)
Observations	106,744	106,744	106,744	106,744	106,744	106,744
R-squared	0.051	0.07	0.051	0.07	0.099	0.122
City FE	No	Yes	No	Yes	Yes	Yes
Year FE	No	No	Yes	Yes	Yes	No
Industry FE	No	No	No	No	Yes	No
Industry-Year FE	No	No	No	No	No	Yes

Note: Robust standard deviations are provided in parentheses.

*** p<0.01, ** p<0.05, * p<0.1

OA-C.3 Policy Effects on Alternative Measures of Firm Performance

Tables OA-9–OA-11 present the policy effects on intensive margin of R&D expenditure (conditional on RDX > 0), revenue, and labor productivity, respectively.

Table OA-9: The Effect of Policy on R&D Expenditure Conditional on RDX > 0

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Dependent Variable	ln(RDX)	ln(RDX)	ln(RDX)	ln(RDX)	ln(RDX)	ln(RDX)
Pilot · After	0.0497*	0.1567***	0.0508*	0.1588***	0.1908***	0.1866***
	(0.0281)	(0.0287)	(0.0281)	(0.0286)	(0.0279)	(0.0283)
Observations	120,101	120,101	120,101	120,101	120,101	120,101
R-squared	0.216	0.246	0.219	0.249	0.318	0.334
City FE	No	Yes	No	Yes	Yes	Yes
Year FE	No	No	Yes	Yes	Yes	No
Industry FE	No	No	No	No	Yes	No
Industry-Year FE	No	No	No	No	No	Yes

Note: Robust standard deviations are provided in parentheses.

*** p<0.01, ** p<0.05, * p<0.1

Table OA-10: The Effect of Policy on Firm-level Revenue

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Dependent Variable	ln(Revenue)	ln(Revenue)	ln(Revenue)	ln(Revenue)	ln(Revenue)	ln(Revenue)
Pilot · After	0.6906***	0.6172***	0.6808***	0.6074***	0.5554***	0.5703***
	(0.0079)	(0.0078)	(0.0079)	(0.0078)	(0.0076)	(0.0076)
Observations	1,251,210	1,251,210	1,251,210	1,251,210	1,251,210	1,251,210
R-squared	0.014	0.075	0.017	0.079	0.173	0.191
City FE	No	Yes	No	Yes	Yes	Yes
Year FE	No	No	Yes	Yes	Yes	No
Industry FE	No	No	No	No	Yes	No
Industry-Year FE	No	No	No	No	No	Yes

Note: Robust standard deviations are provided in parentheses.

*** p<0.01, ** p<0.05, * p<0.1

Table OA-11: The Effect of Policy on Firm-level Labor Productivity

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Dependent Variable	$\ln\left(\frac{\text{Rev}}{\text{Labor}}\right)$	$\ln\left(\frac{\text{Rev}}{\text{Labor}}\right)$	$\ln\left(\frac{\text{Rev}}{\text{Labor}}\right)$	$\ln\left(\frac{\text{Rev}}{\text{Labor}}\right)$	$\ln\left(\frac{\text{Rev}}{\text{Labor}}\right)$	$\ln\left(\frac{\text{Rev}}{\text{Labor}}\right)$
Pilot · After	0.3724***	0.3004***	0.3574***	0.2857***	0.2734***	0.2848***
	(0.0053)	(0.0052)	(0.0052)	(0.0052)	(0.0049)	(0.0049)
Observations	1,250,855	1,250,855	1,250,855	1,250,855	1,250,855	1,250,855
R-squared	0.018	0.073	0.026	0.082	0.222	0.234
City FE	No	Yes	No	Yes	Yes	Yes
Year FE	No	No	Yes	Yes	Yes	No
Industry FE	No	No	No	No	Yes	No
Industry-Year FE	No	No	No	No	No	Yes

Note: Robust standard deviations are provided in parentheses. In the dependent variable row, "Rev" denotes Revenue.

*** p<0.01, ** p<0.05, * p<0.1

OA-C.4 Robustness Check— National Intellectual Property Demonstration Cities

Policy Background. China's National Intellectual Property Demonstration City is a place-based policy designation granted by the central IP authority—formerly the State Intellectual Property Office (SIPO) and, following the 2018 reform, the China National Intellectual Property Administration (CNIPA).[49] According to official program documents, the designation is intended to promote city-level improvements in the full IP governance chain, including IP creation, utilization/commercialization, protection/enforcement, management, and public services. Demonstration cities are typically required to formulate a multi-year implementation plan with quantified objectives and to participate in periodic evaluations under a dynamic management framework, creating incentives for sustained upgrading of local IP institutions rather than one-off policy effort. We compile a list of all demonstration cities designated between 2012 and 2019, as shown in Table OA-12. Of the eight cities in the 2012–2013 patent-backed lending policy batches, two (Quanzhou and Weifang) also appear in the concurrent National Intellectual Property Demonstration City lists.

[49] See the recent announcement by CNIPA at `https://www.cnipa.gov.cn/art/2024/1/11/art_75_189636.html`.

Table OA-12: National Intellectual Property Demonstration Cities (2012–2019)

Batch (Date)	Admin. Level	Cities / Districts
1st Batch (Apr 2012)	Sub-provincial	Wuhan, Guangzhou, Shenzhen, Chengdu, Hangzhou, Jinan, Qingdao, Harbin, Nanjing, Dalian, Xi'an
	Prefecture-level	Changsha, Suzhou, Nantong, Zhenjiang, Zhengzhou, Luoyang, Dongying, Yantai, Fuzhou, Quanzhou, Wenzhou, Wuhu
2nd Batch (Aug 2013)	Sub-provincial	Xiamen, Ningbo, Changchun
	Prefecture-level	Dongguan, Wuxi, Zhuzhou, Taizhou, Weifang, Zibo, Hefei, Jiaxing, Nanyang, Huzhou, Changji, Xinxiang, Guiyang
	County-level	Changshu, Kunshan
3rd Batch (Mar 2015)	Prefecture-level	Changzhou, Anyang, Yichang, Xiangtan, Panzhihua, Foshan, Zhongshan, Chaoyang (Beijing), Nanchang
	County-level	Jiangyin, Danyang, Zhangjiagang
4th Batch (May 2016)	Prefecture-level	Mianyang, Huizhou, Deyang, Haidian (Beijing), Minhang (Shanghai), Xiqing (Tianjin), Jiangbei (Chongqing)
	County-level	Jimo, Haimen, Ningguo, Yiwu
5th Batch (May 2018)	Prefecture-level	Ma'anshan, Shantou, Shijiazhuang, Xuzhou, Jiulongpo (Chongqing), Shenyang
6th Batch (May 2019)	Prefecture-level	Pudong (Shanghai), Kunming, Yancheng, Jinhua, Nanning, Zhuhai, Binhai (Tianjin)

Notes: The table includes municipal districts (e.g., Haidian, Pudong) and county-level cities as specified in the official batches from the CNIPA (China National Intellectual Property Administration).

Empirical Specification To isolate the potential confounding effect from the National Intellectual Property Demonstration City policy, we augment our baseline DID specification by including an additional interaction term between a dummy variable for demonstration cities and the post-treatment period indicator. Specifically, we specify the following model for the firm-level analysis:

$$
\begin{aligned}
Y_{i,t} = &\beta_0 + \beta_1 \cdot \text{Pilot}_{i,t} \cdot \text{After}_{i,t} + \beta_2 \cdot \text{Demo}_{i,t} \cdot \text{After}_{i,t} \\
&+ \beta_3 \text{Pilot}_{i,t} + \beta_4 \text{After}_{i,t} + \beta_5 \text{Demo}_{i,t} + f_i + f_{s(i),t} + \nu_{i,t}
\end{aligned} \tag{OA-4}
$$

and the following model for the city-sector-level analysis:

$$
\begin{aligned}
Y_{c,s,t} = &\beta_0 + \beta_1 \cdot \text{Pilot}_{c,s,t} \cdot \text{After}_{c,s,t} + \beta_2 \cdot \text{Demo}_{c,s,t} \cdot \text{After}_{c,s,t} \\
&+ \beta_3 \text{Pilot}_{c,s,t} + \beta_4 \text{After}_{c,s,t} + \beta_5 \text{Demo}_{c,s,t} + f_c + f_{s,t} + \nu_{c,s,t}
\end{aligned} \tag{OA-5}
$$

where $\text{Demo}_{i,t}$ ($\text{Demo}_{c,s,t}$) is a dummy variable indicating whether firm i (city-sector (c,s)) is located in a National Intellectual Property Demonstration City at time t. The other variables are defined as in the main text. We use the SAT firm-level data from 2011 to 2014 to estimate these models. Therefore, we focus on the first two batches of demonstration cities designated in 2012 and 2013.

Results Table OA-13 presents the estimation results after controlling for the potential confounding effect from the National Intellectual Property Demonstration City policy. The coefficient estimates for the main treatment effect, Pilot · After, remain largely consistent with our findings reported in Table A.1, suggesting that our main results are robust to accounting for the demonstration city policy.

Table OA-13: The Effects of Policy: Controlling National IP Demonstration Cities

	Model 1	Model 2	Model 3	Model 4
Dependent Variable	$\mathbb{I}_{\text{RDX}>0}$	$\ln(\text{RDX}+1)$	$\ln(\text{RTFP})$	$\sigma^{\text{mrpk}}_{c,s,t}$
Pilot · After	0.0138**	0.1082**	0.0323**	-0.1290**
	(0.0068)	(0.0540)	(0.0153)	(0.0600)
Demo · After	-0.0109***	-0.0709***	-0.0179***	-0.0477*
	(0.0033)	(0.0262)	(0.0067)	(0.0284)
Observations	135,946	135,946	73,105	16,251
R-squared	0.658	0.695	0.928	0.153
Firm FE	Yes	Yes	Yes	No
City FE	No	No	No	Yes
Industry-Year FE	Yes	Yes	Yes	Yes

Note: Robust standard deviations are provided in parentheses. *** p<0.01, ** p<0.05, * p<0.1

OA-C.5 Policy Effect on City-Level Wage

Table OA-14 presents the policy effects on city-level wages.

Table OA-14: The Effect of Policy on City-level Wage

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Dependent Variable	Wage	Wage	Wage	Wage	Wage	Wage
Pilot · After	0.0046	0.0057	0.0059	0.0077	0.0195	-0.0197
	(0.0453)	(0.0333)	(0.0400)	(0.0237)	(0.0297)	(0.0426)
Observations	1,616	1,616	1,616	1,616	1,494	1,094
R-squared	0.788	0.904	0.829	0.945	0.960	0.973
City FE	No	Yes	No	Yes	Yes	Yes
Year FE	No	No	Yes	Yes	Yes	No
Industry FE	No	No	No	No	Yes	No
Industry-Year FE	No	No	No	No	No	Yes

Note: Robust standard deviations are provided in parentheses.
*** p<0.01, ** p<0.05, * p<0.1

OA-C.6 Placebo Test

To verify that our main results are not driven by spurious correlations, chance, or unobserved time-varying trends, we conduct a non-parametric permutation test (placebo test) following the methodology of **?** and **?**. The procedure randomizes the assignment of the "pilot" status across cities while maintaining the temporal structure of the policy implementation and the sample composition. The specific algorithm is implemented as follows:

1. **Calibration:** We first record the actual number of pilot cities in our baseline sample, denoted as K.

2. **Random Permutation:** We perform a simulation with $M = 1,000$ iterations. In each iteration m:

   (a) We define the pool of eligible cities for randomization.[50]

[50] To ensure that the counterfactual treatment units are comparable in scale to the actual pilot cities and

(b) We assign a random number to every city (or synthetic city cluster) in the base year.

(c) We sort cities by this random variable and select the top K cities to form a counterfactual "placebo treatment group." The remaining cities serve as the control group.

(d) This placebo status is fixed for a given city across all time periods.

3. **Estimation:** Using the generated placebo treatment group, we construct a false difference-in-differences estimator. We estimate the following regression model, which mirrors our preferred baseline specification:

$$Y_{i,t} = \beta_0 + \beta_m^{placebo} \cdot \text{PlaceboPilot}_i^m \times \text{After}_{i,t} + \beta_2 \text{Pilot}_i + \beta_3 \text{After}_{i,t} + f_{c(i)} + f_{s(i),t} + v_{i,t}, \tag{OA-6}$$

or

$$\sigma_{c,s,t}^{\text{mrpk}} = \beta_0 + \beta_m^{placebo} \cdot \text{PlaceboPilot}_{c,s}^m \times \text{After}_{c,s,t} + \beta_2 \text{Pilot}_{c,s} + \beta_3 \text{After}_{c,s,t} + f_c + f_{s,t} + v_{c,s,t}, \tag{OA-7}$$

where:

- where $Y_{i,t}$ denotes one of three main outcome variables: (1) $\mathbb{1}(\text{RDX}_{i,t} > 0)$ for whether the firm conducts R&D investment, (2) $\ln(\text{RDX}_{i,t} + 1)$ for the scale of R&D investment, and (3) $\ln(\text{RTFP}_{i,t})$ for firm's revenue-based productivity. .
- PlaceboPilot_c^m is a binary indicator equal to 1 if city c is randomly assigned as a pilot in iteration m.

4. **Distribution and Inference:** We store the estimated coefficient $\hat{\beta}_m^{placebo}$ for each of the 1,000 simulations. We then construct the kernel density estimate of these placebo coefficients. The statistical significance of the true policy effect is assessed by comparing the actual estimate, $\hat{\beta}^{true}$, against this empirical distribution. The two-sided p-value is calculated as the proportion of placebo estimates that exceed the true estimate in absolute magnitude:

$$p = \frac{1}{M} \sum_{m=1}^{M} \mathbf{1} \left(|\hat{\beta}_m^{placebo}| > |\hat{\beta}^{true}| \right)$$

to mitigate small-sample noise, we aggregate non-pilot cities with fewer than 500 firms into 20 synthetic city clusters prior to the randomization procedure.

If the policy effect is identified correctly, the distribution of $\hat{\beta}_m^{placebo}$ should be centered around zero, and $\hat{\beta}^{true}$ should lie in the extreme tails of the distribution.

OA-D Proofs

This section provides self-contained proofs for the firm's optimization problem and the comparative statics of policy changes. Throughout the analysis, we assume that wages and interest rates remain fixed.

OA-D.1 Model

We lay out the firm's dynamic problem, technology, financial constraints, and the law of motion for productivity. The wage w and the interest rate r are taken as exogenous. Without altering the core fundamentals, we refine the R&D decision-making framework by introducing both discrete and continuous choice variables.

State variables and controls. The firm's state at the beginning of the period is (a, ϕ), where $a \geq a_{\min} > 0$ denotes net worth and $\phi > \underline{\phi}$ denotes firm productivity. The firm chooses next period net worth $a' \geq a_{\min}$, consumption (or payouts) $c \geq 0$, a discrete R&D participation decision $I \in \{0, 1\}$, and a continuous R&D intensity choice $\mu \geq 0$.

Preferences. Per-period utility is CRRA,

$$U(c) = \begin{cases} \dfrac{c^{1-\epsilon}}{1-\epsilon}, & \epsilon \geq 0, \ \epsilon \neq 1, \\ \\ \ln c, & \epsilon = 1, \end{cases}$$

and the firm discounts the future at rate $\beta \in (0, 1)$.

Dynamic problem. Given the fixed wage, the firm's value function satisfies the Bellman equation

$$V(a, \phi) = \max_{I \in \{0,1\},\, a' \geq a_{\min},\, \mu \geq 0,\, c \geq 0} \Big\{ U(c) + \beta\, \mathbb{E}\big[V(a', \phi') \mid \phi, I, \mu\big] \Big\}, \tag{OA-8}$$

subject to the period budget constraint

$$c + a' + \chi_\mu(\mu; \phi) + I \cdot \chi_f(\phi) = y(a, \phi), \tag{OA-9}$$

and the productivity process described below.

Productivity dynamics. Productivity evolves according to

$$\ln(\phi') = \rho \ln(\phi) + \bar{\mu} + I \cdot \mu + \sigma_\xi \xi', \tag{OA-10}$$

where $0 < \rho < 1$, $\bar{\mu}$ is a constant, and $\sigma_\xi > 0$. The stochastic term $\xi_{i,t+1}$ follows a truncated normal distribution, rescaled to have $\mathbb{E}[\xi_{i,t+1}] = 0$ and $\mathrm{Var}(\xi_{i,t+1}) = 1$. Its support is bounded: for some $\bar{\xi} > 0$, $|\xi_{i,t+1}| \leq \bar{\xi}$ a.s.

Resources and static profit maximization. Per-period resources are given by

$$y(a, \phi) = \pi(a, \phi) + (1 + r)a, \tag{OA-11}$$

where operating profits $\pi(a, \phi)$ are determined by a static optimization problem:

$$\pi(a, \phi) = \max_{k,l} \left\{ (\phi^{\alpha_\phi} k^{\alpha_k} l^{\alpha_l})^{\frac{\sigma-1}{\sigma}} - w \cdot l - (r + \delta) \cdot k \right\}, \tag{OA-12}$$

subject to the borrowing constraint

$$k \leq \theta a + \eta \Psi(\phi). \tag{OA-13}$$

with $\theta \geq 1$, $\eta \geq 0$, and $\Psi(\cdot) \geq 0$ strictly increasing (i.e., $\Psi'(\cdot) > 0$). Here k denotes capital and l denotes labor, w is the wage, and $r + \delta$ is the sum of capital's rental and depreciation rates..

Parameter restrictions. We impose $\beta(1 + r) \leq 1$.

OA-D.2 Capital Choice and Capital's Shadow Price

Firms solve the profit maximization problem embedded in (OA-12) to determine their optimal choices of capital and labor, given their net worth a and productivity ϕ. The (constrained) optimal capital and labor choices are:

$$k^*(a,\, \phi;\, w) = \Gamma(w) M(a, \phi; w)^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}, \tag{OA-14}$$

$$l^*(a,\, \phi;\, w) = \frac{\tilde{\alpha}_l}{\tilde{\alpha}_k w} \Gamma(w) M(a, \phi; w)^{-\frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \tag{OA-15}$$

where $\tilde{\alpha}_\ell = \frac{\sigma-1}{\sigma}\alpha_\ell$, $\ell \in \{\phi, l, k\}$ and $\Gamma(w) = \left(\frac{\tilde{\alpha}_k^{1-\tilde{\alpha}_l}\tilde{\alpha}_l^{\tilde{\alpha}_l}}{w^{\tilde{\alpha}_l}}\right)^{\frac{1}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}$, M is the shadow price of capital or the marginal revenue product of capital (MRPK hereafter) given as:

$$M(a,\phi;w) \ = \ \max\left\{r+\delta, \ \frac{\Gamma(w)^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}}\phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_l}}}{[\theta a+\eta\Psi(\phi)]^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}}}\right\}. \tag{OA-16}$$

Proof of Equation (OA-14)-(OA-16). The firm's optimization problem is

$$\max_{l,k}\left\{(\phi^{\alpha_\phi}k^{\alpha_k}l^{\alpha_l})^{\frac{\sigma-1}{\sigma}} - wl - (r+\delta)k\right\}$$
$$s.t. \ k \le \theta a + \eta\Psi(\phi).$$

Using the scaled output elasticities $\tilde{\alpha}_\ell = \frac{\sigma-1}{\sigma}\alpha_\ell$, the revenue function is $R = \phi^{\tilde{\alpha}_\phi}k^{\tilde{\alpha}_k}l^{\tilde{\alpha}_l}$. Let λ denote the Lagrange multiplier on the capital constraint. The Lagrangian is:

$$\mathcal{L} = \phi^{\tilde{\alpha}_\phi}k^{\tilde{\alpha}_k}l^{\tilde{\alpha}_l} - wl - (r+\delta)k + \lambda(\theta a + \eta\Psi(\phi) - k)$$

The first-order conditions (FOCs) for labor and capital are:

$$\frac{\partial\mathcal{L}}{\partial l} = \tilde{\alpha}_l\frac{R}{l} - w = 0 \implies l = \frac{\tilde{\alpha}_l R}{w}, \tag{OA-17}$$
$$\frac{\partial\mathcal{L}}{\partial k} = \tilde{\alpha}_k\frac{R}{k} - (r+\delta+\lambda) = 0. \tag{OA-18}$$

Let $M \equiv r+\delta+\lambda$ denote the shadow price of capital. From (OA-18), we have $R = \frac{Mk}{\tilde{\alpha}_k}$. Substituting this into (OA-17) gives the optimal labor-capital ratio:

$$l = \frac{\tilde{\alpha}_l}{\tilde{\alpha}_k w}Mk. \tag{OA-19}$$

Now, substitute (OA-19) back into the revenue function expression $R = \frac{Mk}{\tilde{\alpha}_k}$:

$$\begin{aligned}
\frac{Mk}{\tilde{\alpha}_k} &= \phi^{\tilde{\alpha}_\phi}k^{\tilde{\alpha}_k}\left(\frac{\tilde{\alpha}_l Mk}{\tilde{\alpha}_k w}\right)^{\tilde{\alpha}_l} \\
M\tilde{\alpha}_k^{-1}k &= \phi^{\tilde{\alpha}_\phi}k^{\tilde{\alpha}_k+\tilde{\alpha}_l}M^{\tilde{\alpha}_l}\left(\frac{\tilde{\alpha}_l}{\tilde{\alpha}_k w}\right)^{\tilde{\alpha}_l} \\
M^{1-\tilde{\alpha}_l}k^{1-\tilde{\alpha}_k-\tilde{\alpha}_l} &= \phi^{\tilde{\alpha}_\phi}\tilde{\alpha}_k^{1-\tilde{\alpha}_l}\left(\frac{\tilde{\alpha}_l}{w}\right)^{\tilde{\alpha}_l} \\
k^{1-\tilde{\alpha}_k-\tilde{\alpha}_l} &= \phi^{\tilde{\alpha}_\phi}M^{-(1-\tilde{\alpha}_l)}\tilde{\alpha}_k^{1-\tilde{\alpha}_l}\left(\frac{\tilde{\alpha}_l}{w}\right)^{\tilde{\alpha}_l}
\end{aligned}$$

Solving for k yields the optimal capital demand as a function of the shadow price M:

$$k^* = \left(\frac{\tilde{\alpha}_k^{1-\tilde{\alpha}_l}\tilde{\alpha}_l^{\tilde{\alpha}_l}}{w^{\tilde{\alpha}_l}}\right)^{\frac{1}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} M^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}\phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} = \Gamma(w)M^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}\phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}.$$

This matches Equation (OA-14). Substituting k^* into (OA-19) gives l^*:

$$\begin{aligned} l^* &= \frac{\tilde{\alpha}_l}{\tilde{\alpha}_k w}M\left[\Gamma(w)M^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}\phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}\right] \\ &= \frac{\tilde{\alpha}_l}{\tilde{\alpha}_k w}\Gamma(w)M^{-\frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}\phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}. \end{aligned}$$

This matches Equation (OA-15).

To find the shadow price M, we consider the constraint. If the constraint is not binding ($\lambda = 0$), then $M = r + \delta$. If the constraint is binding ($\lambda > 0$), then $k = \theta a + \eta\Psi(\phi)$. We invert the $k^*(M)$ equation:

$$\begin{aligned} k &= \Gamma(w)M^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}\phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \\ M^{\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} &= \frac{\Gamma(w)\phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}}{k} \\ M &= \left(\frac{\Gamma(w)\phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}}{k}\right)^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}} = \frac{\Gamma(w)^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}}\phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_l}}}{k^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}}}. \end{aligned}$$

Substituting the binding capital limit $k = \theta a + \eta\Psi(\phi)$, we obtain the expression in (OA-16). $\square$

Given the model setting, the per-period resource function is

$$\begin{aligned} y(a,\phi) &= \left[\frac{1-\tilde{\alpha}_l}{\tilde{\alpha}_k}M(a,\phi;w) - (r+\delta)\right]k^*(a,\phi;w) + (1+r)a \\ &= \Gamma(w)\phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}\left[\frac{1-\tilde{\alpha}_l}{\tilde{\alpha}_k}M(a,\phi;w)^{-\frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} - (r+\delta)M(a,\phi;w)^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}\right] \\ &\quad + (1+r)a \qquad \text{(OA-20)} \end{aligned}$$

where $M(a,\phi;w)$ is the shadow price of capital defined in (OA-16) and $\Gamma(w)$ is the constant defined in the capital choice decision.

Proof of Equation (OA-20). Recall that per-period resources are the sum of operating profits and the gross return on net worth: $y(a,\phi) = \pi(a,\phi) + (1+r)a$.

From the first-order conditions, we established that operating profit can be expressed as:

$$\pi(a,\phi) = \left[\frac{1-\tilde{\alpha}_l}{\tilde{\alpha}_k} M - (r+\delta)\right] k^*. \tag{OA-21}$$

Substituting the optimal capital choice $k^* = \Gamma(w) M^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}$ into this expression:

$$\begin{aligned}
\pi(a,\phi) &= \left[\frac{1-\tilde{\alpha}_l}{\tilde{\alpha}_k} M - (r+\delta)\right] \left(\Gamma(w) M^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}\right) \\
&= \Gamma(w) \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \left[\frac{1-\tilde{\alpha}_l}{\tilde{\alpha}_k} M \cdot M^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} - (r+\delta) M^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}\right].
\end{aligned} \tag{OA-22}$$

We simplify the exponent of M in the first term and substitute this back into the profit equation:

$$\pi(a,\phi) = \Gamma(w) \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \left[\frac{1-\tilde{\alpha}_l}{\tilde{\alpha}_k} M^{-\frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} - (r+\delta) M^{-\frac{1-\tilde{\alpha}_l}{\Delta}}\right].$$

Finally, adding the gross return on net worth yields the result:

$$y(a,\phi) = \Gamma(w) \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \left[\frac{1-\tilde{\alpha}_l}{\tilde{\alpha}_k} M^{-\frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} - (r+\delta) M^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}}\right] + (1+r)a.$$

□

To facilitate the subsequent analysis, we introduce $\hat{y}(a,\phi) = y(a,\phi) - a_{\min}$, which inherits all relevant properties of $y(a,\phi)$.

OA-D.3 Proof of Proposition 1

Proof of Proposition 1. Let $k^u(\phi)$ denote the unconstrained capital demand, which corresponds to the shadow price $M = r+\delta$. From Equation (OA-14), we have $k^u(\phi) \propto \phi^{\upsilon_\phi}$ with $\upsilon_\phi := \frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}$, implying $\frac{d\log k^u(\phi)}{d\log\phi} = \upsilon_\phi$.

We first establish and subsequently examine the properties of the threshold $\hat{a}(\phi)$ the properties of $M(a,\phi)$ in the binding region (i.e., where $a < \hat{a}(\phi)$).

Proposition of $\hat{a}(\phi)$. The threshold $\hat{a}(\phi)$ is defined as the net worth where the firm's borrowing capacity exactly equals this unconstrained demand:

$$k^u(\phi) = \theta\hat{a}(\phi) + \eta\Psi(\phi). \tag{OA-23}$$

Let $\varsigma_{\hat{a}(\phi)} := \frac{d\log \hat{a}(\phi)}{d\log \phi}$ and $\varsigma_{\Psi}(\phi) := \frac{d\log \Psi(\phi)}{d\log \phi}$. To find the elasticity $\varsigma_{\hat{a}(\phi)}$, we differentiate (OA-23) with respect to ϕ:

$$\frac{dk^u(\phi)}{d\phi} = \theta\frac{d\hat{a}(\phi)}{d\phi} + \eta\frac{d\Psi(\phi)}{d\phi}.$$

Multiplying the equation by ϕ allows us to express terms in elasticities (using the identity $x\frac{df}{dx} = f \cdot \frac{d\log f}{d\log x}$):

$$k^u(\phi)\upsilon_\phi = \theta\hat{a}(\phi)\varsigma_{\hat{a}(\phi)} + \eta\Psi(\phi)\varsigma_\Psi(\phi).$$

Solving for $\varsigma_{\hat{a}(\phi)}$:

$$\varsigma_{\hat{a}(\phi)} = \frac{k^u(\phi)\upsilon_\phi - \eta\Psi(\phi)\varsigma_\Psi(\phi)}{\theta\hat{a}(\phi)}. \tag{OA-24}$$

Factoring out $\frac{k^u(\phi)}{\theta\hat{a}(\phi)}$ from (OA-24) yields (20).

To determine the sign, we return to equation (OA-24). Using the binding condition $\theta\hat{a}(\phi) = k^u(\phi) - \eta\Psi(\phi)$, we can rewrite the elasticity as:

$$\varsigma_{\hat{a}(\phi)} = \frac{k^u(\phi)\upsilon_\phi - \eta\Psi(\phi)\varsigma_\Psi(\phi)}{k^u(\phi) - \eta\Psi(\phi)}.$$

For the threshold to be economically meaningful (i.e., $\hat{a}(\phi) > 0$), the unconstrained capital must exceed the collateral value, $k^u(\phi) > \eta\Psi(\phi)$, ensuring the denominator is positive.

Under Assumption 2, we have $\upsilon_\phi > \varsigma_\Psi(\phi)$ for all ϕ. Since $k^u(\phi) > \eta\Psi(\phi) > 0$, it follows that:

$$k^u(\phi)\upsilon_\phi > k^u(\phi)\varsigma_\Psi(\phi) > \eta\Psi(\phi)\varsigma_\Psi(\phi).$$

Thus, the numerator is strictly positive, implying $\varsigma_{\hat{a}(\phi)} > 0$.

Properties of $M(a, \phi)$. In the region where $a < \hat{a}(\phi)$, Equation (OA-16) implies:

$$M(a,\phi) = \frac{\Gamma(w)^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}l}}\phi^{\frac{\tilde{\alpha}\phi}{1-\tilde{\alpha}_l}}}{[\theta a + \eta\Psi(\phi)]^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}}}. \tag{OA-25}$$

An increase in a raises the denominator while leaving the numerator unchanged; thus, $M(a, \phi)$ is strictly decreasing in a within the binding region.

To analyze the sensitivity of $M(a, \phi)$ with respect to ϕ, we take the natural logarithm of Equation (OA-25) and compute the partial derivative $\frac{\partial \ln M}{\partial \ln \phi}$:

$$\frac{\partial \ln M(a,\phi)}{\partial \ln \phi} = \frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_l} - \frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}\frac{\eta\Psi(\phi)}{\theta a + \eta\Psi(\phi)}\frac{\partial \ln \Psi(\phi)}{\partial \ln \phi}$$

$$
\begin{aligned}
&= \frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}\left(\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l} - \frac{\eta\Psi(\phi)}{\theta a+\eta\Psi(\phi)}\frac{\partial \ln \Psi(\phi)}{\partial \ln \phi}\right) \\
&\geq \frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}\left(\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l} - \frac{\partial \ln \Psi(\phi)}{\partial \ln \phi}\right) \\
&= \frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}\left(\frac{d\log k^u(\phi)}{d\log \phi} - \frac{\partial \ln \Psi(\phi)}{\partial \ln \phi}\right) \\
&> 0,
\end{aligned}
$$

where the inequality holds because $0 \leq \frac{\eta\Psi(\phi)}{\theta a+\eta\Psi(\phi)} < 1$ and the final strict inequality follows from Assumption 2.

□

OA-D.4 Optimal Choices of R&D

We first establish Lemmas 2 through 12 in Section OA-D.4.1. Subsequently, we provide the proof of Proposition 2 in Section OA-D.4.2.

OA-D.4.1 Lemmas for the Proof of Proposition 2

Lemma 2 (Continuous and Increasing Per-Period Resources Function)**.** The profit function $\pi(a,\,\phi)$ is continuous, weakly increasing in net worth a, and strictly increasing in productivity ϕ. Consequently, the per-period resources function $y(a,\,\phi)$ is continuous and strictly increasing in both arguments.

Proof of Lemma 2. We first establish continuity. The profit maximization problem is

$$
\pi(a,\phi) = \max_{k,l}\left\{\left(\phi^{\alpha_\phi}k^{\alpha_k}l^{\alpha_l}\right)^{\frac{\sigma-1}{\sigma}} - wl - (r+\delta)k\right\}
$$

subject to $0 \leq k \leq \theta a + \eta\Psi(\phi)$ and $l \geq 0$. The objective function is continuous in (k, l, a, ϕ). The constraint correspondence $\Gamma(a,\phi) = \{(k,l) \in \mathbb{R}^2_+ : k \leq \theta a + \eta\Psi(\phi)\}$ is non-empty, compact-valued, and continuous (both upper and lower hemi-continuous) with respect to (a,ϕ), given $\Psi(\phi) \geq 0$ and $\Psi'(\phi) > 0$. By Berge's Theorem of the Maximum (see **?**, or **?** Theorem 3.6), the value function $\pi(a,\phi)$ is continuous in (a,ϕ).

Next, we analyze monotonicity using the Envelope Theorem for constrained optimization problems. Let $\lambda(a,\phi) \geq 0$ be the Lagrange multiplier associated with the borrowing constraint

$k \leq \theta a + \eta\Psi(\phi)$. The Lagrangian is:

$$\mathcal{L} = (\phi^{\alpha_\phi} k^{\alpha_k} l^{\alpha_l})^{\frac{\sigma-1}{\sigma}} - wl - (r+\delta)k + \lambda(a,\phi)[\theta a + \eta\Psi(\phi) - k].$$

Monotonicity in a: By the Envelope Theorem,

$$\frac{\partial \pi(a,\phi)}{\partial a} = \frac{\partial \mathcal{L}}{\partial a} = \theta\lambda(a,\phi). \tag{OA-26}$$

Since $\lambda(a,\phi) \geq 0$ (strictly positive when constrained and zero when unconstrained), $\frac{\partial \pi}{\partial a} \geq 0$. Thus, $\pi(a,\phi)$ is weakly increasing in a.

Monotonicity in ϕ: Similarly,

$$\frac{\partial \pi(a,\phi)}{\partial \phi} = \frac{\partial \mathcal{L}}{\partial \phi} = \underbrace{\frac{\sigma-1}{\sigma}\alpha_\phi \phi^{\alpha_\phi \frac{\sigma-1}{\sigma}-1}(k^{\alpha_k} l^{\alpha_l})^{\frac{\sigma-1}{\sigma}}}_{\text{Direct productivity effect}} + \underbrace{\lambda(a,\phi)\eta\Psi'(\phi)}_{\text{Constraint relaxation effect}},$$

where the direct effect is strictly positive for positive k and l (due to the Cobb-Douglas-like structure with $\alpha_\phi > 0$), and the relaxation effect is non-negative ($\lambda \geq 0$, $\eta \geq 0$, $\Psi'(\phi) > 0$). Therefore, $\frac{\partial \pi}{\partial \phi} > 0$.

Properties of $y(a,\phi)$: The per-period resource function is $y(a,\phi) = \pi(a,\phi) + (1+r)a$.

1. **Continuity:** Follows from continuity of $\pi(a,\phi)$ and linearity of $(1+r)a$.

2. **Monotonicity:**

$$\begin{aligned}
\frac{\partial y}{\partial a} &= \frac{\partial \pi}{\partial a} + (1+r) \geq 1+r > 0, \\
\frac{\partial y}{\partial \phi} &= \frac{\partial \pi}{\partial \phi} > 0.
\end{aligned}$$

Thus, $y(a,\phi)$ is strictly increasing in both arguments. □

Lemma 3 (Supermodularity of Per-Period Resources Function)**.** The profit function $\pi(a, \phi)$ is globally concave with respect to net worth a. Regarding the interaction of net worth and productivity:

(i) For unconstrained firms ($a \geq \hat{a}(\phi)$), the cross-partial derivative is zero: $\frac{\partial^2 \pi}{\partial a \partial \phi} = 0$.

(ii) For constrained firms ($a_{\min} \leq a < \hat{a}(\phi)$), $\pi(a,\phi)$ is strictly supermodular (i.e., $\frac{\partial^2 \pi}{\partial a \partial \phi} > 0$) if Assumption 2 holds.

The per-period resources function $y(a,\, \phi)$ (and $\hat{y}(a,\, \phi)$) inherits these properties.

Proof of Lemma 3. We analyze the second-order properties of $\pi(a, \phi)$ using the shadow price of capital, $M(a, \phi)$. Recall from Equation (OA-26) that in the constrained region, the marginal value of net worth is $\frac{\partial \pi}{\partial a} = \theta[M(a, \phi) - (r + \delta)]$.

1. Concavity in Net Worth (a): Differentiating the marginal value with respect to a:

$$\frac{\partial^2 \pi}{\partial a^2} = \theta \frac{\partial M(a, \phi)}{\partial a}.$$

From Equation (OA-16), $M(a, \phi)$ is strictly decreasing in the capital stock $k = \theta a + \eta \Psi(\phi)$. Since k is linear in a, $\frac{\partial M}{\partial a} < 0$ for $a_{\min} \leq a < \hat{a}(\phi)$. For $a \geq \hat{a}(\phi)$, $M = r + \delta$ is constant, so $\frac{\partial^2 \pi}{\partial a^2} = 0$. Thus, $\pi_{aa} \leq 0$ globally. Since $y(a, \phi)$ is linear in a relative to π, $y_{aa} = \pi_{aa} \leq 0$.

2. Supermodularity (a, ϕ): For $a_{\min} \leq a < \hat{a}(\phi)$, the cross-partial derivative is:

$$\frac{\partial^2 \pi}{\partial a \partial \phi} = \theta \frac{\partial M(a, \phi)}{\partial \phi}.$$

Thus, $\pi_{a\phi} > 0$ if and only if $\frac{\partial M}{\partial \phi} > 0$. It is convenient to analyze the log-derivative of M. Taking logs of the constrained case in Equation (OA-16):

$$\ln M(a, \phi) = \text{const} + \frac{1 - \tilde{\alpha}_k - \tilde{\alpha}_l}{1 - \tilde{\alpha}_l} \left(\upsilon_\phi \ln \phi - \ln[\theta a + \eta \Psi(\phi)] \right),$$

where we used the definition $\upsilon_\phi \equiv \frac{\tilde{\alpha}_\phi}{1 - \tilde{\alpha}_k - \tilde{\alpha}_l}$. Differentiating with respect to ϕ:

$$\frac{\partial \ln M}{\partial \phi} \propto \frac{\upsilon_\phi}{\phi} - \frac{\eta \Psi'(\phi)}{\theta a + \eta \Psi(\phi)}.$$

Multiplying by ϕ, the condition $\frac{\partial M}{\partial \phi} > 0$ is equivalent to:

$$\upsilon_\phi - \frac{\eta \Psi(\phi)}{\theta a + \eta \Psi(\phi)} \frac{\phi \Psi'(\phi)}{\Psi(\phi)} > 0 \qquad \Longleftrightarrow \qquad \upsilon_\phi > s(a, \phi) \varsigma_\Psi(\phi),$$

where $\varsigma_\Psi(\phi) = \frac{d \log \Psi(\phi)}{d \log \phi}$, and $s(a, \phi) \equiv \frac{\eta \Psi(\phi)}{\theta a + \eta \Psi(\phi)} \in [0, 1)$ is the share of collateral in total borrowing capacity.

Under Assumption 2, $\upsilon_\phi > \varsigma_\Psi(\phi)$. Since $s(a, \phi) < 1$, we have:

$$\upsilon_\phi > \varsigma_\Psi(\phi) > s(a, \phi) \varsigma_\Psi(\phi).$$

Therefore, under the model primitives, $\frac{\partial M}{\partial \phi} > 0$ and consequently $\frac{\partial^2 \pi}{\partial a \partial \phi} > 0$ for all constrained states. $\square$

Lemma 4 (Existence and Uniqueness of Value Function)**.** Let $\upsilon_\phi = \frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}$. The value function $V(a,\phi)$ exists, is unique, and belongs to the weighted space $\mathcal{C}_\omega$ defined by the weight function $\omega(a,\phi) = 1 + a^{1-\epsilon} + \kappa\phi^{\upsilon_\phi(1-\epsilon)}$ with $\epsilon \neq 1$ and $\omega(a,\phi) = 1 + |\ln(a)| + \kappa\upsilon_\phi|\ln(\phi)|$ with $\epsilon = 1$, for some constant $\kappa > 0$.

Proof of Lemma 4. Fix $\epsilon \geq 0$ and let

$$\upsilon_\phi := \frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}.$$

We work on

$$\mathcal{S} := \{(a,\phi) : a \geq a_{\min},\ \phi \geq \underline{\phi}\}, \qquad a_{\min} > 0 \text{ and } \underline{\phi} > 0,$$

and impose the same lower bound on next period net worth $a' \geq a_{\min}$.[51]

Let $\mathcal{C}_\omega$ be the space of continuous functions $v : \mathcal{S} \to \mathbb{R}$ endowed with the weighted sup norm

$$\|v\|_\omega := \sup_{(a,\phi)\in\mathcal{S}} \frac{|v(a,\phi)|}{\omega(a,\phi)},$$

where

$$\omega(a,\phi) := \begin{cases} 1 + a^{1-\epsilon} + \kappa\,\phi^{\upsilon_\phi(1-\epsilon)}, & \epsilon \neq 1, \\ \\ 1 + |\ln a| + \kappa\,\upsilon_\phi|\ln\phi|, & \epsilon = 1, \end{cases}$$

for a constant $\kappa > 0$ chosen below. Consider the Bellman operator

$$(Tv)(a,\phi) = \max_{\substack{I\in\{0,1\},\ \mu\geq 0,\\ a'\geq a_{\min},\ c\geq 0}} \left\{U(c) + \beta\,\mathbb{E}\big[v(a',\phi') \mid \phi, I, \mu\big]\right\},$$

subject to

$$c + a' + \chi_v(\mu;\phi) + I\cdot\chi_f(\phi) = y(a,\phi), \qquad \ln\phi' = \rho\ln\phi + \bar{\mu} + I\cdot\mu + \sigma_\xi\xi'.$$

Step 1 (reward is ω-bounded). Feasibility implies $c \leq y(a,\phi)$, so $U(c) \leq U(y(a,\phi))$ when U is increasing (all CRRA/log cases). From the static input choice problem (OA-22), unconstrained profits scale as ϕ^{υ_ϕ}; since constrained profits are bounded above by unconstrained profits, there exists $C_\pi > 0$ such that

$$\pi(a,\phi) \leq C_\pi\,\phi^{\upsilon_\phi} \qquad \forall (a,\phi)\in\mathcal{S}.$$

[51] Given the distribution ξ' is bounded, and $\mu \geq 0$, $\underline{\phi}$ exists.

Therefore,

$$y(a,\phi) = (1+r)a + \pi(a,\phi) \le (1+r)a + C_\pi \phi^{\upsilon_\phi}. \tag{OA-27}$$

Moreover, since $a \ge a_{\min}$ and $\phi \ge \underline{\phi}$, there exists $\underline{y} := (1+r)a_{\min} + \pi(a_{\min}, \underline{\phi}) > 0$ such that

$$y(a,\phi) \ge \underline{y} > 0 \qquad \forall (a,\phi) \in \mathcal{S}. \tag{OA-28}$$

We claim that $U \circ y \in \mathcal{C}_\omega$, i.e., there exists $B_u < \infty$ such that

$$\sup_{(a,\phi)\in\mathcal{S}} \frac{|U(y(a,\phi))|}{\omega(a,\phi)} \le B_u. \tag{OA-29}$$

We verify this by cases.

Case $\epsilon = 0$. Here $U(c) = c$, hence by (OA-27),

$$\frac{|U(y(a,\phi))|}{\omega(a,\phi)} \le \frac{(1+r)a + C_\pi \phi^{\upsilon_\phi}}{1 + a + \kappa \phi^{\upsilon_\phi}} \le (1+r) + \frac{C_\pi}{\kappa}.$$

Case $0 < \epsilon < 1$. Here $U(c) = \frac{c^{1-\epsilon}}{1-\epsilon}$ and $1-\epsilon \in (0,1)$, so $(x_1+x_2)^{1-\epsilon} \le x_1^{1-\epsilon} + x_2^{1-\epsilon}$ for $x_1, x_2 \ge 0$. Using (OA-27),

$$\begin{aligned}
\frac{|U(y(a,\phi))|}{\omega(a,\phi)} &= \frac{y(a,\phi)^{1-\epsilon}}{(1-\epsilon)\,\omega(a,\phi)} \\
&\le \frac{\big((1+r)a + C_\pi \phi^{\upsilon_\phi}\big)^{1-\epsilon}}{(1-\epsilon)\,\omega(a,\phi)} \\
&\le \frac{(1+r)^{1-\epsilon} a^{1-\epsilon} + C_\pi^{1-\epsilon} \phi^{\upsilon_\phi(1-\epsilon)}}{(1-\epsilon)\left(1 + a^{1-\epsilon} + \kappa \phi^{\upsilon_\phi(1-\epsilon)}\right)} \\
&\le \frac{(1+r)^{1-\epsilon}}{1-\epsilon} + \frac{C_\pi^{1-\epsilon}}{(1-\epsilon)\kappa}.
\end{aligned}$$

Case $\epsilon = 1$. Here $U(c) = \ln c$ and (OA-28) guarantees $y(a,\phi) > 0$. Using the elementary inequality that for all $x_1, x_2 > 0$,

$$|\ln(x_1 + x_2)| \le 1 + |\ln x_1| + |\ln x_2|,$$

we take $x_1 = (1+r)a$ and $x_2 = C_\pi \phi^{\upsilon_\phi}$ to obtain

$$\begin{aligned}
|\ln(y(a,\phi))| &\le \left|\ln\left((1+r)a + C_\pi \phi^{\upsilon_\phi}\right)\right| \\
&\le 1 + |\ln((1+r)a)| + |\ln(C_\pi \phi^{\upsilon_\phi})| \\
&\le 1 + |\ln(1+r)| + |\ln a| + |\ln C_\pi| + \upsilon_\phi |\ln \phi|.
\end{aligned}$$

Therefore, dividing by $\omega(a,\phi) = 1 + |\ln a| + \kappa v_\phi |\ln \phi|$ yields

$$\sup_{(a,\phi)\in\mathcal{S}} \frac{|U(y(a,\phi))|}{\omega(a,\phi)} \le 1 + |\ln(1+r)| + |\ln C_\pi| + \frac{1}{\kappa} < \infty,$$

with an explicit bound depending only on r, C_π, κ, v_ϕ.

Case $\epsilon > 1$. Here $U(c) = \frac{c^{1-\epsilon}}{1-\epsilon} < 0$ and

$$|U(y(a,\phi))| = \frac{y(a,\phi)^{1-\epsilon}}{\epsilon - 1}.$$

Since $1-\epsilon < 0$ and $y(a,\phi) \ge \underline{y}$, we have $y(a,\phi)^{1-\epsilon} \le \underline{y}^{1-\epsilon}$, hence

$$\frac{|U(y(a,\phi))|}{\omega(a,\phi)} \le \frac{\underline{y}^{1-\epsilon}}{(\epsilon-1)\,\omega(a,\phi)} \le \frac{\underline{y}^{1-\epsilon}}{\epsilon - 1}.$$

This verifies (OA-29).

Step 2 (drift bound for the discounted weight). Define the drift ratio

$$\mathcal{R}(a,\phi) := \beta\, \frac{\sup_{I,\mu,a',c} \mathbb{E}[\omega(a',\phi') \mid a,\phi]}{\omega(a,\phi)}.$$

Let (I,μ,a',c) be any feasible choice at (a,ϕ). From feasibility and $c \ge 0$,

$$a_{\min} \le a' \le y(a,\phi). \tag{OA-30}$$

Moreover, feasibility implies $\chi_v(\mu;\phi) \le y(a,\phi)$, hence (since $\chi_v(\cdot;\phi)$ is increasing)

$$0 \le \mu \le \mu_{\max}(a,\phi) := \chi_v^{-1}(y(a,\phi);\phi). \tag{OA-31}$$

Finally,

$$\phi' = \exp\bigl(\rho \ln\phi + \bar{\mu} + I\cdot\mu + \sigma_\xi \xi'\bigr) = \phi^\rho\, \exp(\bar{\mu} + I\cdot\mu)\, \exp(\sigma_\xi \xi').$$

Because ξ' is bounded, its exponential moments are finite. In particular, for any $q \in \mathbb{R}$ there exists a finite constant

$$M_\xi(q) := \mathbb{E}\bigl[e^{q\sigma_\xi \xi'}\bigr] < \infty.$$

We now verify that for a suitable $\kappa > 0$ there exist $\lambda \in (0,1)$ and $b < \infty$ such that

$$\beta \sup_{I,\mu,a',c} \mathbb{E}[\omega(a',\phi') \mid a,\phi] \le \lambda\, \omega(a,\phi) + b, \qquad \forall (a,\phi) \in \mathcal{S}. \tag{OA-32}$$

(Equivalently, $\mathcal{R}(a,\phi)$ is eventually bounded by some $\lambda < 1$ outside a compact set.)

Case $\epsilon = 0$. Here $\omega(a,\phi) = 1 + a + \kappa\phi^{\upsilon_\phi}$. Using (OA-30) and monotonicity,

$$\mathbb{E}[a' \mid a, \phi] \le y(a,\phi),$$

and using (OA-31),

$$\begin{aligned}\mathbb{E}[\phi'^{\upsilon_\phi} \mid a,\phi] &= \phi^{\rho\upsilon_\phi}\, e^{\upsilon_\phi\bar{\mu}}\, \mathbb{E}\Big[e^{\upsilon_\phi I\mu}\, e^{\upsilon_\phi\sigma_\xi\xi'}\Big] \\ &\le \phi^{\rho\upsilon_\phi}\, e^{\upsilon_\phi\bar{\mu}}\, e^{\upsilon_\phi\mu_{\max}(a,\phi)}\, M_\xi(\upsilon_\phi).\end{aligned}$$

Hence for a constant $C_\phi := e^{\upsilon_\phi\bar{\mu}} M_\xi(\upsilon_\phi)$,

$$\sup_{I,\mu,a',c} \mathbb{E}[\omega(a',\phi') \mid a,\phi] \le 1 + y(a,\phi) + \kappa C_\phi\, \phi^{\rho\upsilon_\phi}\, e^{\upsilon_\phi\mu_{\max}(a,\phi)}.$$

Combining this with (OA-27) yields an explicit upper bound for $\mathcal{R}(a,\phi)$. Under the conditions,

$$\lim_{a\to\infty} \frac{e^{\upsilon_\phi\mu_{\max}(a,\phi)}}{a} = 0 \quad (\text{for fixed } \phi), \qquad \limsup_{\phi\to\infty} \frac{e^{\mu_{\max}(a,\phi)}}{\phi^{1-\rho}} = 0 \quad (\text{for fixed } a),$$

we obtain

$$\limsup_{a\to\infty} \mathcal{R}(a,\phi) \le \beta(1+r), \qquad \limsup_{\phi\to\infty} \mathcal{R}(a,\phi) \le \beta\,\frac{C_\pi}{\kappa}.$$

Thus, if $\beta(1+r) < 1$ and $\kappa > C_\pi$, then $\limsup_{(a,\phi)\to\infty} \mathcal{R}(a,\phi) < 1$, which implies (OA-32) for some $\lambda \in (0,1)$ and $b < \infty$.

Case $0 < \epsilon < 1$. Here $\omega(a,\phi) = 1 + a^{1-\epsilon} + \kappa\phi^{\upsilon_\phi(1-\epsilon)}$. Using (OA-30) and that $t \mapsto t^{1-\epsilon}$ is increasing and concave,

$$(a')^{1-\epsilon} \le y(a,\phi)^{1-\epsilon} \le (1+r)^{1-\epsilon} a^{1-\epsilon} + C_\pi^{1-\epsilon}\phi^{\upsilon_\phi(1-\epsilon)}.$$

Moreover, letting $q := \upsilon_\phi(1-\epsilon) > 0$ and using (OA-31),

$$\begin{aligned}\mathbb{E}[\phi'^{q} \mid a,\phi] &= \phi^{\rho q}\, e^{q\bar{\mu}}\, \mathbb{E}\Big[e^{qI\mu}\, e^{q\sigma_\xi\xi'}\Big] \\ &\le \phi^{\rho q}\, e^{q\bar{\mu}}\, e^{q\mu_{\max}(a,\phi)}\, M_\xi(q).\end{aligned}$$

Hence for $C_\phi := e^{q\bar{\mu}} M_\xi(q)$,

$$\sup_{I,\mu,a',c} \mathbb{E}[\omega(a',\phi') \mid a,\phi] \le 1 + (1+r)^{1-\epsilon}a^{1-\epsilon} + C_\pi^{1-\epsilon}\phi^{\upsilon_\phi(1-\epsilon)} + \kappa C_\phi\, \phi^{\rho\upsilon_\phi(1-\epsilon)} e^{\upsilon_\phi(1-\epsilon)\mu_{\max}(a,\phi)}.$$

Under the conditions,

$$\lim_{a\to\infty} \frac{e^{\upsilon_\phi\mu_{\max}(a,\phi)}}{a} = 0 \quad (\text{for fixed } \phi), \qquad \limsup_{\phi\to\infty} \frac{e^{\mu_{\max}(a,\phi)}}{\phi^{1-\rho}} = 0 \quad (\text{for fixed } a),$$

we obtain

$$\limsup_{a\to\infty} \mathcal{R}(a,\phi) \le \beta(1+r)^{1-\epsilon}, \qquad \limsup_{\phi\to\infty} \mathcal{R}(a,\phi) \le \beta\,\frac{C_\pi^{1-\epsilon}}{\kappa}.$$

Thus, if $\beta(1+r)^{1-\epsilon} < 1$ (it holds under $\beta(1+r) \le 1$) and $\kappa > C_\pi^{1-\epsilon}$, then (OA-32) holds.

Case $\epsilon = 1$. Here $\omega(a,\phi) = 1 + |\ln a| + \kappa v_\phi |\ln \phi|$. From (OA-30) and monotonicity of $\ln(\cdot)$ on $(0,\infty)$,

$$|\ln a'| \le \max\{|\ln a_{\min}|,\ |\ln y(a,\phi)|\}.$$

As in Step 1, using $|\ln(x_1+x_2)| \le 1 + |\ln x_1| + |\ln x_2|$ with $x_1 = (1+r)a$ and $x_2 = C_\pi \phi^{v_\phi}$,

$$|\ln y(a,\phi)| \le C_0 + |\ln a| + v_\phi |\ln \phi|, \qquad C_0 := 1 + |\ln(1+r)| + |\ln C_\pi|.$$

For the productivity component,

$$|\ln \phi'| = \left|\rho \ln \phi + \bar{\mu} + I\cdot\mu + \sigma_\xi \xi'\right| \le \rho|\ln \phi| + |\bar{\mu}| + \mu_{\max}(a,\phi) + |\sigma_\xi|\,|\xi'|.$$

Taking conditional expectations and using boundedness of ξ' gives

$$\mathbb{E}[|\ln \phi'| \mid a,\phi] \le \rho|\ln\phi| + C_1 + \mu_{\max}(a,\phi)$$

for some constant $C_1 < \infty$.

To conclude the drift inequality (OA-32), it suffices to assume the inverse-cost bound is sub-logarithmic in both arguments:

$$\lim_{a\to\infty} \frac{\mu_{\max}(a,\phi)}{\ln a} = 0 \quad (\text{for fixed } \phi), \qquad \limsup_{\phi\to\infty} \frac{\mu_{\max}(a,\phi)}{\ln \phi} = 0 \quad (\text{for fixed } a).$$

Under these conditions, the $\mu_{\max}$ term is asymptotically negligible relative to $\omega(a,\phi)$, and we obtain

$$\limsup_{a\to\infty} \mathcal{R}(a,\phi) \le \beta, \qquad \limsup_{\phi\to\infty} \mathcal{R}(a,\phi) \le \beta\,\rho + \frac{\beta}{\kappa}\cdot(\text{constant}).$$

Since $\beta < 1$ and $\beta\rho < 1$, choosing κ sufficiently large yields (OA-32).

Case $\epsilon > 1$. Now $1-\epsilon < 0$, so $\omega(a,\phi) = 1 + a^{1-\epsilon} + \kappa\phi^{v_\phi(1-\epsilon)}$ is bounded above and below by positive constants on $\mathcal{S}$:

$$1 \le \omega(a,\phi) \le 1 + a_{\min}^{1-\epsilon} + \kappa\,\underline{\phi}^{v_\phi(1-\epsilon)} < \infty.$$

Hence $\|\cdot\|_\omega$ is equivalent to the usual sup norm on $\mathcal{S}$. Moreover, by (OA-28) and $\epsilon > 1$, the one-period utility $U(c)$ is bounded above by 0 and bounded below by $U(\underline{y})$ on feasible choices (since $c \leq y(a,\phi)$ and $y(a,\phi) \geq \underline{y}$). Therefore T is a standard β-contraction on bounded continuous functions, and thus also a contraction on $(\mathcal{C}_\omega, \|\cdot\|_\omega)$.

Step 3 (existence and uniqueness). Combining Step 1 and Step 2, we have that T maps $\mathcal{C}_\omega$ into itself and satisfies the drift condition (OA-32). By the Weighted Contraction Mapping Theorem (see e.g., **?**, or **?**), T is a contraction on $(\mathcal{C}_\omega, \|\cdot\|_\omega)$, and therefore admits a unique fixed point $V \in \mathcal{C}_\omega$. □

Lemma 5 (Bounded Process)**.** Given the law of motion for productivity

$$\ln(\phi') = \rho \ln(\phi) + \bar{\mu} + I \cdot \mu + \sigma_\xi \xi',$$

where $(\xi'_t)_{t\geq 0}$ are i.i.d. shocks with $|\xi'| \leq \bar{\xi} < \infty$ almost surely. Let $P(\cdot \mid \phi, I, \mu)$ denote the Markov kernel (c.d.f.) of ϕ'. Given $1 > \rho > 0$, the following hold:

(Supp) For each (ϕ, I, μ), the conditional support of ϕ' is the compact interval

$$\mathcal{Z}(\phi, I, \mu) := \big[\underline{\phi}'(\phi, I, \mu),\, \overline{\phi}'(\phi, I, \mu)\big] = \big[e^{\rho \ln \phi + \bar{\mu} + I\cdot\mu - \sigma_\xi \bar{\xi}},\, e^{\rho \ln \phi + \bar{\mu} + I\cdot\mu + \sigma_\xi \bar{\xi}}\big].$$

(MLR) Let $z(\phi, I, \mu) := \rho \ln \phi + \bar{\mu} + I \cdot \mu$. Then $X := \ln \phi' \mid (\phi, I, \mu)$ forms a location family on $[z(\phi, I, \mu) - \sigma_\xi \bar{\xi},\ z(\phi, I, \mu) + \sigma_\xi \bar{\xi}]$ that satisfies the monotone likelihood ratio (MLR) property in x as z increases. Since $x \mapsto e^x$ is strictly increasing, the MLR (and thus strict FOSD) holds for ϕ'.

(F1+) For any (ϕ, μ) with $\mu > 0$, $P(\cdot \mid \phi, 1, \mu)$ strictly FOSD-dominates $P(\cdot \mid \phi, 0, 0)$.

(F2+) For fixed (I, μ), $P(\cdot \mid \phi, I, \mu)$ is strictly FOSD- and MLR-increasing in ϕ.

(Fμ+) For fixed ϕ and $I = 1$, $P(\cdot \mid \phi, 1, \mu)$ is strictly FOSD- and MLR-increasing in μ.

Proof of Lemma 5. Fix $(\phi, I, \mu) \in [\underline{\phi}, \infty) \times \{0, 1\} \times [0, \infty)$ and recall the law of motion

$$\ln(\phi') = \rho \ln(\phi) + \bar{\mu} + I \cdot \mu + \sigma_\xi \xi',$$

where ξ' is a truncated standard normal with support $[-\bar{\xi}, \bar{\xi}]$ almost surely, for some finite $\bar{\xi} > 0$. For notational convenience, define

$$z(\phi, I, \mu) \ :=\ \rho \ln(\phi) + \bar{\mu} + I \cdot \mu.$$

(Supp). Because $\xi' \in [-\bar{\xi}, \bar{\xi}]$ a.s., we have

$$\ln(\phi') = z(\phi, I, \mu) + \sigma_\xi \xi' \in \Big[z(\phi, I, \mu) - \sigma_\xi \bar{\xi},\ z(\phi, I, \mu) + \sigma_\xi \bar{\xi}\Big] \quad \text{a.s.}$$

Exponentiating gives

$$\phi' \ \in \ \Big[\exp\big(z(\phi, I, \mu) - \sigma_\xi \bar{\xi}\big),\ \exp\big(z(\phi, I, \mu) + \sigma_\xi \bar{\xi}\big)\Big] \quad \text{a.s.}$$

Hence, writing

$$\underline{\phi}'(\phi, I, \mu) := e^{z(\phi, I, \mu) - \sigma_\xi \bar{\xi}} \quad \text{and} \quad \overline{\phi}'(\phi, I, \mu) := e^{z(\phi, I, \mu) + \sigma_\xi \bar{\xi}},$$

the conditional support of ϕ' is the compact interval

$$\mathcal{Z}(\phi, I, \mu) = \big[\underline{\phi}'(\phi, I, \mu), \overline{\phi}'(\phi, I, \mu)\big],$$

as claimed.

(MLR). Let ξ' denote a truncated standard normal random variable on $[-\bar{\xi}, \bar{\xi}]$, with density p_ξ and c.d.f. P_ξ, and

$$\ln(\phi') \ = \ z(\phi, I, \mu) + \sigma_\xi \xi'.$$

For each real z we can therefore write

$$X_z \ := \ \ln(\phi') \mid (\phi, I, \mu) \quad \text{with } z = z(\phi, I, \mu)$$

as

$$X_z \ = \ z + \sigma_\xi \xi'.$$

The conditional density of X_z with respect to Lebesgue measure is

$$f(x \mid z) \ = \ \frac{1}{\sigma_\xi}\, p_\xi\!\left(\frac{x - z}{\sigma_\xi}\right) \mathbf{1}\Big\{x \in \big[z - \sigma_\xi \bar{\xi},\ z + \sigma_\xi \bar{\xi}\big]\Big\},$$

where p_ξ does not depend on z. Since ξ' is a truncated standard normal,

$$p_\xi(e) = \frac{\varphi(e)}{\int_{-\bar{\xi}}^{\bar{\xi}} \varphi(u)\, du} \mathbf{1}\{e \in [-\bar{\xi}, \bar{\xi}]\},$$

with $\varphi(e) = (2\pi)^{-1/2} \exp\{-e^2/2\}$ the standard normal density.

Thus, for x in the (nonempty) intersection of the supports of X_{z_1} and X_{z_2}, and for any $z_2 > z_1$,

$$\frac{f(x \mid z_2)}{f(x \mid z_1)} = \frac{p_\xi\left(\frac{x-z_2}{\sigma_\xi}\right)}{p_\xi\left(\frac{x-z_1}{\sigma_\xi}\right)} = \frac{\varphi\left(\frac{x-z_2}{\sigma_\xi}\right)}{\varphi\left(\frac{x-z_1}{\sigma_\xi}\right)} \cdot \frac{\displaystyle\int_{-\bar{\xi}}^{\bar{\xi}} \varphi(u)\,du}{\displaystyle\int_{-\bar{\xi}}^{\bar{\xi}} \varphi(u)\,du} = \frac{\varphi\left(\frac{x-z_2}{\sigma_\xi}\right)}{\varphi\left(\frac{x-z_1}{\sigma_\xi}\right)}.$$

The ratio of the normalizing constants for the truncated distribution cancels because it does not depend on z or x.

Using the explicit form of φ, we have

$$\frac{\varphi\left(\frac{x-z_2}{\sigma_\xi}\right)}{\varphi\left(\frac{x-z_1}{\sigma_\xi}\right)} = \exp\left(-\frac{(x-z_2)^2-(x-z_1)^2}{2\sigma_\xi^2}\right),$$

so that

$$\begin{aligned}
\ln\frac{f(x \mid z_2)}{f(x \mid z_1)} &= -\frac{(x-z_2)^2-(x-z_1)^2}{2\sigma_\xi^2} + \text{constant in } x \\
&= -\frac{1}{2\sigma_\xi^2}\Big[(x^2-2xz_2+z_2^2)-(x^2-2xz_1+z_1^2)\Big] + \text{constant} \\
&= -\frac{1}{2\sigma_\xi^2}\Big[-2x(z_2-z_1)+(z_2^2-z_1^2)\Big] + \text{constant}.
\end{aligned}$$

Differentiating with respect to x yields

$$\frac{d}{dx}\ln\frac{f(x \mid z_2)}{f(x \mid z_1)} = \frac{z_2-z_1}{\sigma_\xi^2} \; > \; 0,$$

since $z_2 > z_1$ and $\sigma_\xi > 0$. Hence the likelihood ratio $f(x \mid z_2)/f(x \mid z_1)$ is strictly increasing in x on the intersection of the supports, and therefore the family $\{X_z\}_{z\in\mathbb{R}}$ satisfies the *monotone likelihood ratio* (MLR) property in x as z increases.[52]

Moreover, since the transformation $x \mapsto e^x$ is strictly increasing, the induced family $\{\phi' = e^{X_z}\}_{z\in\mathbb{R}}$ inherits the MLR ordering: if $z_2 > z_1$, the conditional densities of ϕ' given z_2 and z_1 satisfy the MLR property in ϕ'.

It is well known that MLR ordering implies (strict) first-order stochastic dominance (FOSD) whenever the distributions are not identical (see, e.g., **?**, or **?**). Because X_{z_2} is a non-degenerate right shift of X_{z_1} when $z_2 > z_1$, the dominance is strict. This yields the statement in part (MLR) of the lemma.

[52]See, e.g., **?** for the definition and equivalent characterizations of MLR families.

(F1+), (F2+), and (Fμ+). The properties (F1+), (F2+), and (Fμ+) are immediate corollaries of the MLR result, together with the fact that $z(\phi, I, \mu)$ is strictly increasing in I, ϕ, and μ in the relevant comparisons.

(F1+). Fix (ϕ, μ) with $\mu > 0$ and compare the two regimes $(I, \mu) = (1, \mu)$ and $(I, \mu) = (0, 0)$. By definition,

$$z(\phi, 1, \mu) = \rho \ln \phi + \bar{\mu} + \mu > \rho \ln \phi + \bar{\mu} = z(\phi, 0, 0).$$

Thus the distribution of $\ln(\phi')$ (and hence of ϕ') under $(I, \mu) = (1, \mu)$ strictly MLR- and FOSD-dominates the distribution under $(I, \mu) = (0, 0)$. This is exactly the statement of (F1+).

(F2+). Fix (I, μ) and take $\phi_2 > \phi_1$. Given $0 < \rho < 1$, we have

$$z(\phi_2, I, \mu) - z(\phi_1, I, \mu) = \rho\big(\ln \phi_2 - \ln \phi_1\big) > 0.$$

Therefore the conditional distribution of $\ln(\phi')$ (and thus of ϕ') is strictly MLR- and FOSD-increasing in ϕ, establishing (F2+).

(Fμ+). Finally, fix ϕ and set $I = 1$. For $\mu_2 > \mu_1 \geq 0$,

$$z(\phi, 1, \mu_2) - z(\phi, 1, \mu_1) = \mu_2 - \mu_1 > 0.$$

By the same reasoning, the conditional distribution of $\ln(\phi')$ (and thus of ϕ') is strictly MLR- and FOSD-increasing in μ, proving (Fμ+).

Combining the arguments above verifies all the claims in Lemma 5. $\square$

Lemma 6 (Conditional Concavity and Monotonicity)**.** Let $V^0(a, \phi)$ and $V^1(a, \phi)$ denote the value functions conditional on the discrete choice $I = 0$ (no R&D) and $I = 1$ (positive R&D), respectively.

(i) The global value function $V(a, \phi) = \max\{V^0(a, \phi), V^1(a, \phi)\}$ is strictly increasing in net worth a.

(ii) For each fixed regime $I \in \{0, 1\}$, the conditional value function $V^I(a, \phi)$ is concave in a. Moreover, if $\epsilon > 0$, $V^I(a, \phi)$ is strictly concave in a.

(iii) The optimal consumption policy conditional on regime I, denoted $c^I(a, \phi)$, is increasing in a. Moreover, if $\epsilon > 0$, $c^I(a, \phi)$, is strictly increasing in a.

Proof of Lemma 6. Fix ϕ and a regime $I \in \{0,1\}$. Write the total R&D outlay as

$$\chi(\mu;\phi,I) \; := \; \chi_v(\mu;\phi) + I \cdot \chi_f(\phi),$$

so the within-period feasibility constraint is

$$c + a' + \chi(\mu;\phi,I) \; \le \; y(a,\phi), \qquad c \ge 0, \; a' \ge a_{\min}, \; \mu \ge 0.$$

By Lemma 2, $y(a,\phi)$ is continuous and strictly increasing in a; by Lemma 3, $y(a,\phi)$ is concave in a.

Step 0 (A convenient parametric formulation). For each fixed (ϕ, I), define the value function with *current resources* y as the state:

$$\widetilde{V}^I(y,\phi) \; := \max_{\substack{c \ge 0,\; a' \ge a_{\min},\; \mu \ge 0 \\ c + a' + \chi(\mu;\phi,I) \le y}} \left\{ U(c) + \beta\, \mathbb{E}\big[V(a',\phi') \mid \phi, I, \mu\big] \right\}. \tag{OA-33}$$

Then, by definition of $y(a,\phi)$,

$$V^I(a,\phi) \; = \; \widetilde{V}^I\big(y(a,\phi),\phi\big). \tag{OA-34}$$

This separation is useful because a affects the conditional problem only through the scalar $y(a,\phi)$.

Part (i): $V(a,\phi)$ is strictly increasing in a. Take $a_H > a_L \ge a_{\min}$. Since $y(a,\phi)$ is strictly increasing in a, $y(a_H,\phi) > y(a_L,\phi)$.

Let (c_L, a'_L, μ_L) be an optimal policy for $V^I(a_L,\phi)$ (for either $I = 0$ or $I = 1$). Consider the candidate policy at (a_H,ϕ) that keeps (a',μ) fixed at (a'_L,μ_L) and increases consumption to satisfy the budget:

$$c_{\text{cand}} := y(a_H,\phi) - a'_L - \chi(\mu_L;\phi,I) \; > \; y(a_L,\phi) - a'_L - \chi(\mu_L;\phi,I) = c_L.$$

Thus the candidate is feasible at a_H and delivers strictly higher current utility because U is strictly increasing: $U(c_{\text{cand}}) > U(c_L)$, while continuation utility is unchanged (same (a'_L,μ_L)). Therefore $V^I(a_H,\phi) > V^I(a_L,\phi)$ for each I, and hence

$$V(a,\phi) = \max\{V^0(a,\phi), V^1(a,\phi)\}$$

is strictly increasing in a as the pointwise maximum of two strictly increasing functions.

Part (ii): Concavity (and strict concavity when $\epsilon > 0$) of $V^I(a, \phi)$ in a. We proceed in two substeps.

(a) $\widetilde{V}^I(y, \phi)$ is concave and nondecreasing in y. In (OA-33), the objective does not depend on y; y only appears as the right-hand side of a single linear inequality constraint. The feasible correspondence in (c, a', μ) expands linearly with y. Standard results in concave analysis imply that the value function of a maximization problem is concave and nondecreasing in the resource parameter y whenever the constraint set is convex in the choice variables and the objective is concave in the choice variables.[53]

Moreover, because U is strictly increasing and $a' \geq a_{\min}$, $\mu \geq 0$, the resource constraint binds at any optimum, so $\widetilde{V}^I$ is strictly increasing in y.

If $\epsilon > 0$, then U is strictly concave. In that case, the maximization in (OA-33) yields a unique optimal c for each (y, ϕ, I) (given strict concavity in c and a binding resource constraint), which implies $\widetilde{V}^I(\cdot, \phi)$ is strictly concave in y.[54]

(b) Composition with $y(a, \phi)$. By Lemma 3, $a \mapsto y(a, \phi)$ is concave for each fixed ϕ; by part (a), $y \mapsto \widetilde{V}^I(y, \phi)$ is concave and nondecreasing. Therefore, the composition

$$a \mapsto V^I(a, \phi) = \widetilde{V}^I(y(a, \phi), \phi)$$

is concave in a (composition of a concave nondecreasing function with a concave function). If $\epsilon > 0$, then $\widetilde{V}^I(\cdot, \phi)$ is strictly concave in y, and since $y(a, \phi)$ is strictly increasing in a, the composition is strictly concave in a.

Part (iii): $c^I(a, \phi)$ is increasing (strictly if $\epsilon > 0$) in a. Work with the resource-state formulation (OA-33) and let $\tilde{c}^I(y, \phi)$ denote the optimal consumption as a function of (y, ϕ) in regime I. By (OA-34), the policy of interest is

$$c^I(a, \phi) = \tilde{c}^I\big(y(a, \phi), \phi\big).$$

[53] Here convexity of the constraint set follows from linearity in (c, a') and convexity of $\chi_v(\mu; \phi)$ in μ. Concavity in the choice variables holds because U is concave in c, and (given the induction hypothesis or Lemma 4 plus standard DP arguments) $a' \mapsto \mathbb{E}[V(a', \phi') \mid \phi, I, \mu]$ is concave whenever $V(\cdot, \cdot)$ is concave in a'.

[54] Intuitively, if y changes, optimal c changes; with strictly concave U, this rules out linear segments in the value as a function of y.

Consider the Lagrangian for (OA-33) with multiplier $\lambda \geq 0$ on the resource constraint:

$$\mathcal{L} = U(c) + \beta\,\mathbb{E}\big[V(a',\phi') \mid \phi, I, \mu\big] + \lambda\left(y - c - a' - \chi(\mu;\phi,I)\right).$$

Because U is strictly increasing, the resource constraint binds at the optimum, hence $\lambda > 0$ and the Karush–Kuhn–Tucker (KKT) condition for c is

$$U'(\tilde{c}^I(y,\phi)) = \lambda(y,\phi,I). \tag{OA-35}$$

By the envelope theorem, any optimal multiplier $\lambda(y,\phi,I)$ is a supergradient of the concave function $y \mapsto \widetilde{V}^I(y,\phi)$:

$$\lambda(y,\phi,I) \in \partial_y \widetilde{V}^I(y,\phi).$$

Since $\widetilde{V}^I(\cdot,\phi)$ is concave in y, its supergradients are (set-valued) nonincreasing in y; in particular, any measurable selection $y \mapsto \lambda(y,\phi,I)$ is weakly decreasing in y.

Because U' is strictly decreasing when $\epsilon > 0$ (and weakly decreasing when $\epsilon = 0$), (OA-35) implies that $\tilde{c}^I(y,\phi)$ is weakly increasing in y, and strictly increasing in y when $\epsilon > 0$. Finally, since $y(a,\phi)$ is strictly increasing in a, the composition $c^I(a,\phi) = \tilde{c}^I(y(a,\phi),\phi)$ is weakly increasing in a (and strictly increasing when $\epsilon > 0$).

This proves (i)–(iii). □

Lemma 7 (Increasing Differences of the Flow Objective Function)**.** Fix current productivity ϕ and any feasible continuation choice vector $x = (a',\mu)$. Define the flow objective function (conditional on the discrete choice $I \in \{0,1\}$) as:

$$\mathcal{O}(a,I;x,\phi) \equiv U\Big(y(a,\phi) - \mathcal{E}(I,\mu,a';\phi)\Big) + \beta\mathbb{E}\big[V(a',\phi') \mid \phi, I, \mu\big], \tag{OA-36}$$

where $\mathcal{E}(I,\mu,a';\phi) = a' + \chi_v(\mu;\phi) + I \cdot \chi_f(\phi)$ represents total expenditures.

Then $\mathcal{O}(a,I;x,\phi)$ has *increasing differences* in (a,I): for any $a_H \geq a_L$,

$$\mathcal{O}(a_H,1;x,\phi) - \mathcal{O}(a_H,0;x,\phi) \;\geq\; \mathcal{O}(a_L,1;x,\phi) - \mathcal{O}(a_L,0;x,\phi). \tag{OA-37}$$

If, in addition, U is strictly concave (e.g. CRRA with $\epsilon > 0$, or log utility) and $\chi_f(\phi) > 0$, then the inequality is strict whenever $a_H > a_L$.

Proof of Lemma 7. Fix (ϕ, x). Let $\tilde{n}(a) \equiv y(a,\phi) - a' - \chi_v(\mu;\phi)$, so that consumption under each regime is

$$c^0(a) = \tilde{n}(a), \qquad c^1(a) = \tilde{n}(a) - \chi_f(\phi).$$

Consider the difference in objectives:

$$\begin{aligned}\Delta(a) &:= \mathcal{O}(a,1;x,\phi) - \mathcal{O}(a,0;x,\phi)\\ &= \Big(U(c^1(a)) - U(c^0(a))\Big) \;+\; \beta\Big(\mathbb{E}[V(a',\phi') \mid \phi,1,\mu] - \mathbb{E}[V(a',\phi') \mid \phi,0,\mu]\Big). \end{aligned} \tag{OA-38}$$

The second bracket in (OA-38) depends on (ϕ,μ) but not on current net worth a, and hence is constant in a. Therefore, the monotonicity of $\Delta(a)$ in a is governed entirely by the utility term. Define, for any z such that both arguments are in the domain of U,

$$d(z) := U(z - \chi_f(\phi)) - U(z).$$

Since U is concave and increasing, U' is weakly decreasing, which implies

$$d'(z) = U'(z - \chi_f(\phi)) - U'(z) \;\geq\; 0.$$

Thus $d(\cdot)$ is weakly increasing.

By Lemma 2, $y(a,\phi)$ is strictly increasing in a. Since a' and $\chi_v(\mu;\phi)$ are fixed when we fix x, it follows that $\tilde{n}(a)$ is strictly increasing in a. Hence the composition $d(\tilde{n}(a))$ is weakly increasing in a, so $\Delta(a)$ is weakly increasing in a.

This establishes increasing differences in (a, I) as in (OA-37). If U is strictly concave and $\chi_f(\phi) > 0$, then U' is strictly decreasing and therefore $d'(z) > 0$, so $d(\cdot)$ is strictly increasing; combined with strict monotonicity of $\tilde{n}(a)$, we obtain that $\Delta(a)$ is strictly increasing in a, implying strict increasing differences for $a_H > a_L$. $\square$

Lemma 8 (Monotonicity of the Net Value of Innovation)**.** Fix ϕ and define

$$\Delta(a,\phi) := V^1(a,\phi) - V^0(a,\phi),$$

where $V^I(a,\phi)$ is the value function conditional on the current-period discrete choice $I \in \{0,1\}$. Let[55]

$$\mathcal{A}^1(\phi) := \{a \geq a_{\min} : \; \hat{y}(a,\phi) \geq \chi_f(\phi)\}.$$

[55] Note that $\hat{y}(a,\phi) = y(a,\phi) - a_{\min}$

For $a \notin \mathcal{A}^1(\phi)$, the choice $I = 1$ is infeasible in the current period and hence $V^1(a, \phi) = -\infty$ and $\Delta(a, \phi) = -\infty$.

Then, on $\mathcal{A}^1(\phi)$, $\Delta(a, \phi)$ is weakly increasing in a. Moreover, if U is strictly concave (e.g. CRRA with $\epsilon > 0$ or log utility) and $\chi_f(\phi) > 0$, then $\Delta(a, \phi)$ is strictly increasing in a on $\mathcal{A}^1(\phi)$.

Proof of Lemma 8. Fix ϕ. For any continuation choice $x = (a', \mu)$ with $a' \geq a_{\min}$ and $\mu \geq 0$, define the (conditional) flow objective

$$\mathcal{O}(a, I; x, \phi) := U\Big(y(a, \phi) - a' - \chi_v(\mu; \phi) - I \cdot \chi_f(\phi)\Big) + \beta\, \mathbb{E}\big[V(a', \phi') \mid \phi, I, \mu\big].$$

Let $\widetilde{\mathcal{O}}(a, I; x, \phi)$ be the extension of $\mathcal{O}$ to a common choice set $\mathcal{X} := [a_{\min}, +\infty) \times \mathbb{R}_+$, defined by

$$\widetilde{\mathcal{O}}(a, I; x, \phi) := \begin{cases} \mathcal{O}(a, I; x, \phi), & \text{if } c := y(a, \phi) - a' - \chi_v((\mu; \phi) - I \cdot \chi_f(\phi) \geq 0, \\ -\infty, & \text{otherwise.} \end{cases}$$

Then for each $I \in \{0, 1\}$,

$$V^I(a, \phi) = \sup_{x \in \mathcal{X}} \widetilde{\mathcal{O}}(a, I; x, \phi).$$

If $\hat{y}(a, \phi) < \chi_f(\phi)$, then even at $(a', \mu) = (a_{\min}, 0)$ we have $c = y(a, \phi) - \chi_f(\phi) < 0$, so $\widetilde{\mathcal{O}}(a, 1; x, \phi) = -\infty$ for all $x \in \mathcal{X}$. Hence $V^1(a, \phi) = -\infty$ and $\Delta(a, \phi) = -\infty$.

Step 1: Increasing differences of $\widetilde{\mathcal{O}}$. Fix $x \in \mathcal{X}$ and $\phi > \underline{\phi}$. On the region where $c \geq 0$, Lemma 7 implies that $(a, I) \mapsto \mathcal{O}(a, I; x, \phi)$ has increasing differences in (a, I). On points where $\widetilde{\mathcal{O}} = -\infty$, increasing differences holds trivially. Therefore, for each fixed $x \in \mathcal{X}$, the extended function $(a, I) \mapsto \widetilde{\mathcal{O}}(a, I; x, \phi)$ has increasing differences in (a, I).

Step 2: Supremum preserves increasing differences. Define the indirect payoff

$$F(a, I) := \sup_{x \in \mathcal{X}} \widetilde{\mathcal{O}}(a, I; x, \phi) = V^I(a, \phi).$$

A standard result (see e.g., **?**, or **?**) implies that the pointwise supremum of functions with increasing differences also has increasing differences. Hence $F(a, I)$ has increasing differences in (a, I), i.e., for any $a_H \geq a_L$,

$$F(a_H, 1) - F(a_H, 0) \;\geq\; F(a_L, 1) - F(a_L, 0).$$

Substituting $F(a, I) = V^I(a, \phi)$ yields

$$\Delta(a_H, \phi) \;\geq\; \Delta(a_L, \phi),$$

whenever both sides are finite; in particular, on $\mathcal{A}^1(\phi)$.

If $\epsilon > 0$, for each x the function $\widetilde{\mathcal{O}}(a, I; x, \phi)$ remains strictly supermodular in (a, I) (strictly increasing differences) on the region where both actions are feasible. $F(a, I)$ has *strictly* increasing differences on the set of states where both $I = 0$ and $I = 1$ are feasible and where the supremum is attained at interior choices (which is guaranteed by Lemma 4 and the strict concavity of U). Formally, for any $a_H > a_L$ in $\mathcal{A}^1(\phi)$ (so that choosing $I = 1$ is feasible at both net worth levels) we have

$$F(a_H, 1) - F(a_L, 1) \;>\; F(a_H, 0) - F(a_L, 0). \tag{OA-39}$$

Rearranging terms gives

$$\Delta(a_H, \phi) > \Delta(a_L, \phi) \quad \text{for all } a_H > a_L \text{ in } \mathcal{A}^1(\phi).$$

Thus, $\Delta(a, \phi)$ is strictly increasing in a on $\mathcal{A}^1(\phi)$. For $a \notin \mathcal{A}^1(\phi)$, the choice $I = 1$ is not feasible in the current period, and, by definition, $\Delta(a, \phi) = -\infty$. This completes the proof. □

Lemma 9 (Continuity and Attainment in the Conditional Problems)**.** Fix $(\phi, I) \in [\underline{\phi}, \infty) \times \{0, 1\}$ and define the feasibility domain

$$\mathcal{A}^I(\phi) \;\equiv\; \big\{a \geq a_{\min} :\; \hat{y}(a, \phi) - I \cdot \chi_f(\phi) \geq 0\big\}.$$

For $a \in \mathcal{A}^I(\phi)$, define the conditional value

$$V^I(a, \phi) \;:=\; \max_{\substack{c \geq 0,\; a' \geq a_{\min},\; \mu \geq 0 \\ c + a' + \chi_v(\mu; \phi) + I \cdot \chi_f(\phi) \leq y(a, \phi)}} \big\{U(c) + \beta\, \mathbb{E}\big[V(a', \phi') \mid \phi, I, \mu\big]\big\}.$$

For $a \notin \mathcal{A}^I(\phi)$, set $V^I(a, \phi) := -\infty$. Then:

(i) For every $a \in \mathcal{A}^I(\phi)$, the feasible set

$$\Gamma^I(a, \phi) \;=\; \Big\{(a', \mu) \in [a_{\min}, +\infty) \times \mathbb{R}_+ :\; a' + \chi_v(\mu; \phi) \leq y(a, \phi) - I \cdot \chi_f(\phi)\Big\}$$

is nonempty, compact, and upper hemicontinuous in a.

(ii) The map $(a, a', \mu) \mapsto U\big(y(a,\phi) - a' - \chi_v(\mu;\phi) - I \cdot \chi_f(\phi)\big)$ is continuous. Moreover, for each fixed (ϕ, I),

$$(a', \mu) \mapsto \mathbb{E}\big[V(a', \phi') \mid \phi, I, \mu\big]$$

is continuous on $\mathbb{R}^2_+$.

(iii) For each $a \in \mathcal{A}^I(\phi)$, a maximizer exists and $V^I(\cdot, \phi)$ is continuous on $\mathcal{A}^I(\phi)$. Consequently, for each fixed ϕ, $\Delta(a,\phi) := V^1(a,\phi) - V^0(a,\phi)$ is continuous at all a where both regimes are feasible (i.e., $a \in \mathcal{A}^1(\phi)$), and $\Delta(a,\phi) = -\infty$ for $a \notin \mathcal{A}^1(\phi)$.

Proof of Lemma 9. We proceed in three steps to establish the properties of the constraint correspondence, the objective function, and the value function.

Part (i): Properties of the Correspondence Γ^I. Fix the state (ϕ, I). The correspondence $\Gamma^I : \mathcal{A}^I(\phi) \rightrightarrows [a_{min}, +\infty) \times \mathbb{R}_+$ is defined by:

$$\Gamma^I(a,\phi) = \left\{(a', \mu) \in [a_{min}, +\infty) \times \mathbb{R}_+ : a' + \chi_v(\mu;\phi) \leq y(a,\phi) - I \cdot \chi_f(\phi)\right\}.$$

Non-emptiness: For any $a \in \mathcal{A}^I(\phi)$, the inequality $\hat{y}(a,\phi) - I \cdot \chi_f(\phi) \geq 0$ holds. Since $\chi_v(0;\phi) = 0$ and $a' \geq a_{\min}$ allows for $a' = a_{\min}$, the vector $(a_{\min}, 0)$ is strictly feasible. Thus, $\Gamma^I(a,\phi) \neq \emptyset$.

Compactness: The set is closed as the intersection of the closed sets defined by continuous inequalities. To show boundedness, observe that $a' \leq y(a,\phi)$ implies a' is bounded. For μ, note that $\chi_v(\mu;\phi)$ is convex in μ. Thus, $\lim_{\mu \to \infty} \chi_v(\mu;\phi) = \infty$. Consequently, there exists a $\bar{\mu}$ such that for all feasible choices, $\mu \leq \bar{\mu}$. Therefore, $\Gamma^I(a,\phi)$ is a closed and bounded subset of $\mathbb{R}^2$, hence compact.

Continuity:

1. **Upper Hemicontinuity (u.h.c.):** The constraint function $G(a', \mu; a) = a' + \chi_v(\mu) - [y(a) - I \cdot \chi_f]$ is continuous in all arguments. By the Theorem of the Maximum (constraints section), a correspondence defined by continuous weak inequalities over a compact range is u.h.c.

2. **Lower Hemicontinuity (l.h.c.):** Since $(a_{\min}, 0)$ satisfies the constraint strictly (Slater's condition) for all $a \in \mathcal{A}^I(\phi)$, the correspondence is l.h.c. at every point in the domain.

Thus, Γ^I is continuous.

Part (ii): Joint Continuity of the Objective Function. Define the objective $F(a, a', \mu) \equiv U(y(a,\phi) - I \cdot \chi_f(\phi) - a' - \chi_v(\mu;\phi)) + \beta \mathcal{V}_c^I(a', \mu)$.

Instantaneous Utility: $U(\cdot)$ is continuous on $(0, \infty)$. On the domain defined by the graph of Γ^I, consumption c is a continuous function of (a, a', μ).

Expected Continuation Value:

$$\mathcal{V}_c^I(a', \mu) = \int_{\mathcal{Z}} V(a', e^z) \frac{1}{\sigma_\xi} p_\xi \left(\frac{x - z(\phi, I, \mu)}{\sigma_\xi} \right) dx,$$

where p_ξ is the PDF of ξ', $\mathcal{Z}$ is the support of $\ln(\phi')$ conditional on (ϕ, I, μ) (defined in Lemma 5) and $z(\phi, I, \mu) = \rho \ln \phi + \bar{\mu} + I \cdot \mu$. From Lemma 4, $|V(a', e^z)| \leq C(1 + a'^{(1-\epsilon)} + \kappa e^{z v_\phi (1-\epsilon)})$ (or $|V(a', e^z)| \leq C(1 + |\ln(a')| + \kappa v_\phi |z|)$ if $\epsilon = 1$). The density decays exponentially (e^{-z^2}), which dominates the polynomial/exponential growth of the weight function ω (since v_ϕ is finite). Since the integrand is continuous in (a', μ) for every z, and bounded by an integrable function (the weight function integrated against a truncated normal density with finite moments), the Dominated Convergence Theorem applies. Thus, $\mathcal{V}_c^I(a', \mu)$ is jointly continuous on $[a_{\min}, \infty) \times \mathbb{R}_+$.

Part (iii): Existence and Continuity of V^I. Since Γ^I is non-empty, compact-valued, and continuous, and F is continuous on the graph of Γ^I, Berge's Theorem of the Maximum implies: 1. A solution (a'^*, μ^*) exists. 2. The value function $V^I(a, \phi)$ is continuous in a on $\mathcal{A}^I(\phi)$.

Finally, the difference $\Delta(a, \phi) = V^1(a, \phi) - V^0(a, \phi)$ is the difference of two continuous functions on the intersection of their domains. Since $\chi_f(\phi) > 0$, $\mathcal{A}^1(\phi) \subset \mathcal{A}^0(\phi)$. Thus, $\Delta(a, \phi)$ is continuous for all $a \in \mathcal{A}^1(\phi)$. □

Lemma 10 (Increasing Differences after Optimizing out Savings)**.** Fix $\phi \geq \underline{\phi}$ and set $I = 1$. For $a \geq a_{\min}$ and $\mu \geq 0$, define

$$H(a, \mu; \phi) \;:=\; \max_{a' \geq a_{\min}} \left\{ U\Big(y(a,\phi) - a' - \chi_v(\mu;\phi) - \chi_f(\phi)\Big) \;+\; \beta\, \mathbb{E}\big[V(a', \phi') \mid \phi, 1, \mu\big] \right\},$$

where feasibility is understood via the consumption constraint

$$c \;=\; y(a,\phi) - a' - \chi_v(\mu;\phi) - \chi_f(\phi) \;\geq\; 0,$$

and (only) infeasible choices with $c < 0$ are assigned value $-\infty$.

Then $H(\cdot,\cdot;\phi)$ has *increasing differences* in (a,μ) on the set where it is finite: for any $a_H > a_L$ and $\mu_H > \mu_L \geq 0$ such that all four values $H(a_i,\mu_j;\phi)$ are finite, we have

$$\big[H(a_H,\mu_H;\phi) - H(a_H,\mu_L;\phi)\big] \;\geq\; \big[H(a_L,\mu_H;\phi) - H(a_L,\mu_L;\phi)\big].$$

If, in addition, U is strictly concave on $(0,\infty)$ (e.g. CRRA with $\epsilon > 0$ or log utility), then $H(\cdot,\cdot;\phi)$ has strictly increasing differences on any region where the optimizer implies strictly positive consumption.

Proof of Lemma 10. Fix ϕ, and set $I = 1$. To lighten notation, write

$$y(a) := y(a,\phi), \qquad X_v(\mu) := \chi_v(\mu;\phi), \qquad X_f := \chi_f(\phi),$$

and define

$$\mathcal{V}_c^1(a',\mu) \;:=\; \mathbb{E}\big[V(a',\phi') \mid \phi, 1, \mu\big].$$

For (a,μ,a'), define implied consumption

$$c(a,\mu;a') \;:=\; y(a) - a' - X_v(\mu) - X_f.$$

Now define, for each $a' \geq a_{\min}$,

$$F(a,\mu;a') \;:=\; U\big(c(a,\mu;a')\big) \;+\; \beta\,\mathcal{V}_c^1(a',\mu).$$

Then

$$H(a,\mu;\phi) \;=\; \sup_{a' \geq a_{\min}} F(a,\mu;a').$$

Step 1: For each fixed $a' \geq a_{\min}$, $F(\cdot,\cdot;a')$ has increasing differences in (a,μ) on $\{(a,\mu) : c(a,\mu;a') \geq 0\}$.

Fix $a' \geq a_{\min}$ and restrict attention to (a,μ) such that $c(a,\mu;a') \geq 0$ so that F is differentiable in (a,μ) and coincides with $U(c) + \beta\mathcal{V}^1$. Then

$$\frac{\partial c(a,\mu;a')}{\partial a} = y_a(a) > 0, \qquad \frac{\partial c(a,\mu;a')}{\partial \mu} = -\frac{\partial X_v(\mu)}{\partial \mu} < 0.$$

Since U is concave on $(0,\infty)$ (on $[0,\infty)$ if $\epsilon < 1$), $U''(c) \leq 0$ on $(0,\infty)$ (on $[0,\infty)$ if $\epsilon < 1$). Differentiating,

$$\frac{\partial F(a,\mu;a')}{\partial a} = U'\big(c(a,\mu;a')\big)\,y_a(a),$$

and

$$\begin{aligned}\frac{\partial^2 F(a,\mu;a')}{\partial a\,\partial\mu} &= \frac{\partial}{\partial\mu}\Big(U'\big(c(a,\mu;a')\big)\,y_a(a)\Big)\\ &= U''\big(c(a,\mu;a')\big)\,y_a(a)\,\frac{\partial c(a,\mu;a')}{\partial\mu}\\ &= -U''\big(c(a,\mu;a')\big)\,y_a(a)\,\frac{\partial X_v(\mu)}{\partial\mu}\ \geq\ 0.\end{aligned}$$

Hence $F(\cdot,\cdot;a')$ has increasing differences in (a,μ) wherever $c(a,\mu;a')>0$.

If U is strictly concave on $(0,\infty)$ (on $[0,\infty)$ if $0<\epsilon<1$), then $U''(c)<0$ and the cross-partial is strictly positive on $\{c>0\}$, implying strictly increasing differences for F on that region.

Step 2: The supremum over a' preserves (weak) increasing differences.

By Step 1, each $(a,\mu)\mapsto F(a,\mu;a')$ has increasing differences in (a,μ) on its effective domain (and is $-\infty$ when infeasible). A standard monotone comparative statics result (see e.g., **?**, or **?**) implies that the pointwise supremum

$$H(a,\mu;\phi)=\sup_{a'\geq a_{\min}} F(a,\mu;a')$$

inherits increasing differences in (a,μ) on the set where it is finite. This gives the desired weak inequality.

Step 3 (strictness). When U is strictly concave on $(0,\infty)$, Step 1 yields strict increasing differences for each $F(\cdot,\cdot;a')$ on $\{c>0\}$. Therefore, on any region of (a,μ) where the maximizing choice(s) $a'^*(a,\mu)$ imply strictly positive consumption, the same supremum argument delivers strictly increasing differences for H. □

Lemma 11 (Existence and Monotonicity of the Intensive-Choice Correspondence)**.** Fix ϕ, and set $I=1$. Define the μ-feasibility correspondence

$$\Gamma_\mu(a,\phi)\ :=\ \Big\{\mu\geq 0:\ \hat{y}(a,\phi)-\chi_f(\phi)-\chi_v(\mu;\phi)\geq 0\Big\},$$

where $\hat{y}(a,\phi)=y(a,\phi)-a_{\min}$. For each $a\geq a_{\min}$ such that $\Gamma_\mu(a,\phi)\neq\emptyset$, define the intensive-choice correspondence

$$\mathcal{M}(a,\phi)\ \equiv\ \arg\max_{\mu\in\Gamma_\mu(a,\phi)} H(a,\mu;\phi),$$

where $H(a,\mu;\phi)$ is defined in Lemma 10. Then:

(i) (*Existence and compactness*) For each a with $\Gamma_\mu(a, \phi) \neq \emptyset$, the set $\mathcal{M}(a, \phi)$ is nonempty and compact.

(ii) (*Monotonicity*) The correspondence $\mathcal{M}(\cdot, \phi)$ is nondecreasing in a in the strong set order. In particular, both the minimal and maximal selections from $\mathcal{M}(\cdot, \phi)$ are nondecreasing in a.

Proof of Lemma 11. Fix ϕ, and set $I = 1$.

Step 1: Feasible set for μ is nonempty, compact, and increasing in a. By definition, $\Gamma_\mu(a, \phi) \neq \emptyset$ if and only if $\hat{y}(a, \phi) \geq \chi_f(\phi)$, since $\chi_v(0; \phi) = 0$. When nonempty, $\Gamma_\mu(a, \phi)$ is an interval of the form $[0, \bar{\mu}(a, \phi)]$: indeed, $\chi_v(\cdot; \phi)$ is continuous and strictly increasing in μ and satisfies $\lim_{\mu \to \infty} \chi_v(\mu; \phi) = \infty$, so the inequality $\chi_v(\mu; \phi) \leq \hat{y}(a, \phi) - \chi_f(\phi)$ defines a closed and bounded set. Hence $\Gamma_\mu(a, \phi)$ is compact.

Moreover, by Lemma 2, $y(a, \phi)$ is strictly increasing in a. Therefore, if $a_H > a_L$ and both feasible sets are nonempty, then

$$\hat{y}(a_H, \phi) - \chi_f(\phi) \;\geq\; \hat{y}(a_L, \phi) - \chi_f(\phi),$$

which implies $\Gamma_\mu(a_L, \phi) \subseteq \Gamma_\mu(a_H, \phi)$. Thus $\Gamma_\mu(\cdot, \phi)$ is ascending (nondecreasing) in the strong set order.

Step 2: Continuity of $H(a, \mu; \phi)$ in μ and attainment. Fix a with $\Gamma_\mu(a, \phi) \neq \emptyset$ and fix $\mu \in \Gamma_\mu(a, \phi)$. Given (a, μ, ϕ), the set of feasible a' in the inner problem defining H is

$$\Gamma_{a'}(a, \mu; \phi) \;=\; \Big\{a' \geq a_{\min} : \; a' \leq y(a, \phi) - \chi_f(\phi) - \chi_v(\mu; \phi)\Big\},$$

which is nonempty and compact because $\mu \in \Gamma_\mu(a, \phi)$.

By Lemma 9, the mapping

$$(a', \mu) \mapsto U\Big(y(a, \phi) - a' - \chi_v(\mu; \phi) - \chi_f(\phi)\Big) \;+\; \beta\, \mathbb{E}\big[V(a', \phi') \mid \phi, 1, \mu\big]$$

is continuous on the feasible region, and the feasible correspondence $\Gamma_{a'}(a, \mu; \phi)$ is compact-valued and continuous in μ (in fact, it is an interval with an endpoint that moves continuously in μ). Hence, by Berge's Maximum Theorem, $H(a, \mu; \phi)$ is continuous in μ on $\Gamma_\mu(a, \phi)$, and the maximizer in a' exists.

Since $H(a,\cdot;\phi)$ is continuous on the nonempty compact set $\Gamma_\mu(a,\phi)$, Weierstrass' theorem implies that $\mathcal{M}(a,\phi)$ is nonempty and compact.

Step 3: Monotonicity of $\mathcal{M}(a,\phi)$. By Lemma 10, the function $H(\cdot,\cdot;\phi)$ has increasing differences in (a,μ) on the region where it is finite (in particular, on the feasible set induced by Γ_μ). Combining: (i) increasing differences of the objective, and (ii) the fact that $\Gamma_\mu(\cdot,\phi)$ is ascending in a (Step 1), the Monotone Comparative Statics result (see e.g., **?**, or **?**) implies that the argmax correspondence $\mathcal{M}(\cdot,\phi)$ is nondecreasing in a in the strong set order.

Finally, since each $\mathcal{M}(a,\phi)$ is nonempty and compact (Step 2), its minimal and maximal selections are well-defined, and monotonicity of the correspondence implies that both selections are nondecreasing in a. $\square$

Lemma 12 (Monotonicity on the Interior)**.** Fix ϕ, and set $I=1$. For each $a \geq a_{\min}$ with $\Gamma_\mu(a,\phi) \neq \emptyset$, let

$$\mathcal{M}(a,\phi) \;\equiv\; \arg\max_{\mu\in\Gamma_\mu(a,\phi)} H(a,\mu;\phi),$$

where $H(a,\mu;\phi)$ is defined in Lemma 10. Define the extremal selections

$$\underline{\mu}(a,\phi) := \min \mathcal{M}(a,\phi), \qquad \overline{\mu}(a,\phi) := \max \mathcal{M}(a,\phi).$$

Let $a_H > a_L$.

(i) **(Weak monotonicity; any $\epsilon \geq 0$).** Both $\underline{\mu}(\cdot,\phi)$ and $\overline{\mu}(\cdot,\phi)$ are nondecreasing in a. In particular, for any $\mu_L \in \mathcal{M}(a_L,\phi)$ there exists $\mu_H \in \mathcal{M}(a_H,\phi)$ such that $\mu_H \geq \mu_L$ (e.g. take $\mu_H = \overline{\mu}(a_H,\phi)$).

(ii) **(No downward shift when $\epsilon > 0$).** If $\epsilon > 0$ (CRRA strictly concave) and $\mu_L \in \mathcal{M}(a_L,\phi)$ with $\mu_L > 0$, then every $\mu_H \in \mathcal{M}(a_H,\phi)$ satisfies $\mu_H \geq \mu_L$.

(iii) **(Strict increase from an interior maximizer when $\epsilon > 0$).** Assume $\epsilon > 0$ and there exists a joint maximizer (a'_L,μ_L) of the *two-dimensional* conditional problem at (a_L,ϕ) such that $\mu_L > 0$ and the associated consumption is strictly positive,

$$c_L \;=\; y(a_L,\phi) - a'_L - \chi_v(\mu_L;\phi) - \chi_f(\phi) \;>\; 0.$$

Then every $\mu_H \in \mathcal{M}(a_H,\phi)$ satisfies $\mu_H > \mu_L$.

Proof of Lemma 12. Fix ϕ, and set $I = 1$.

(i) By Lemma 11, the correspondence $\mathcal{M}(\cdot, \phi)$ is nondecreasing in a in the strong set order; hence the minimal and maximal selections $\underline{\mu}(\cdot, \phi)$ and $\overline{\mu}(\cdot, \phi)$ are nondecreasing. For any $\mu_L \in \mathcal{M}(a_L, \phi)$, we have $\mu_L \leq \overline{\mu}(a_L, \phi) \leq \overline{\mu}(a_H, \phi)$, so taking $\mu_H = \overline{\mu}(a_H, \phi) \in \mathcal{M}(a_H, \phi)$ yields $\mu_H \geq \mu_L$.

(ii) Let $\epsilon > 0$. Take any $\mu_H \in \mathcal{M}(a_H, \phi)$ and suppose, for contradiction, that $\mu_H < \mu_L$. Since $\epsilon > 0$ implies U is strictly concave, Lemma 10 implies that $H(\cdot, \cdot; \phi)$ has *strictly* increasing differences in (a, μ) on any region where induced consumption is strictly positive. (Here $H(a_L, \mu_L; \phi)$ and $H(a_L, \mu_H; \phi)$ are finite because μ_L, μ_H are chosen from argmax sets.)

Because μ_L is optimal at a_L,

$$H(a_L, \mu_L; \phi) \;\geq\; H(a_L, \mu_H; \phi) \quad \Rightarrow \quad H(a_L, \mu_L; \phi) - H(a_L, \mu_H; \phi) \;\geq\; 0.$$

Strict increasing differences with $a_H > a_L$ and $\mu_L > \mu_H$ yields

$$H(a_H, \mu_L; \phi) - H(a_H, \mu_H; \phi) \;>\; H(a_L, \mu_L; \phi) - H(a_L, \mu_H; \phi) \;\geq\; 0,$$

hence $H(a_H, \mu_L; \phi) > H(a_H, \mu_H; \phi)$, contradicting $\mu_H \in \mathcal{M}(a_H, \phi)$. Therefore $\mu_H \geq \mu_L$ for every $\mu_H \in \mathcal{M}(a_H, \phi)$.

(iii) Let $\epsilon > 0$ and let (a'_L, μ_L) be a joint maximizer at (a_L, ϕ) with $\mu_L > 0$ and $c_L > 0$ as stated. Define, for fixed a',

$$\widetilde{H}(a, \mu; a', \phi) \;:=\; U\Big(y(a, \phi) - a' - \chi_v(\mu; \phi) - \chi_f(\phi)\Big) \;+\; \beta\, \mathbb{E}\big[V(a', \phi') \mid \phi, 1, \mu\big].$$

Since (a'_L, μ_L) is a joint maximizer and $\mu_L > 0$ with $c_L > 0$, μ_L is an *interior* maximizer of $\mu \mapsto \widetilde{H}(a_L, \mu; a'_L, \phi)$, and thus satisfies the first-order condition

$$\frac{\partial}{\partial \mu} \widetilde{H}(a_L, \mu_L; a'_L, \phi) = -U'(c_L)\, \frac{\partial \chi_v(\mu_L; \phi)}{\partial \mu} \;+\; \beta\, \frac{\partial}{\partial \mu} \mathbb{E}\big[V(a'_L, \phi') \mid \phi, 1, \mu\big]\Big|_{\mu = \mu_L} = 0.$$

Now evaluate the same a' and μ at the higher net worth $a_H > a_L$. Holding $(a', \mu) = (a'_L, \mu_L)$ fixed, consumption is

$$\widetilde{c}_H = y(a_H, \phi) - a'_L - \chi_v(\mu_L; \phi) - \chi_f(\phi) \;>\; y(a_L, \phi) - a'_L - \chi_v(\mu_L; \phi) - \chi_f(\phi) = c_L,$$

because $y(\cdot, \phi)$ is strictly increasing in a (Lemma 2). With $\epsilon > 0$, U is strictly concave, so U' is strictly decreasing and therefore $U'(\widetilde{c}_H) < U'(c_L)$. The continuation term in $\widetilde{H}$ depends on

(a', μ) but *not* on a once a' is held fixed. Hence

$$\frac{\partial}{\partial \mu}\widetilde{H}(a_H, \mu_L; a'_L, \phi) = -U'(\widetilde{c}_H)\,\frac{\partial \chi_v(\mu_L;\phi)}{\partial \mu} \;+\; \beta\,\frac{\partial}{\partial \mu}\mathbb{E}\big[V(a'_L, \phi') \mid \phi, 1, \mu\big]\Big|_{\mu=\mu_L} \;>\; 0,$$

where the inequality follows by comparing to the FOC at a_L (the second term is identical, while the first term becomes strictly larger because $U'(\widetilde{c}_H) < U'(c_L)$ and $\partial \chi_v/\partial \mu > 0$ for $\mu > 0$).

Therefore, there exists $\delta > 0$ small enough such that $\widetilde{H}(a_H, \mu_L{+}\delta; a'_L, \phi) > \widetilde{H}(a_H, \mu_L; a'_L, \phi)$. Using $H(a, \mu; \phi) = \max_{a' \geq a_{\min}} \widetilde{H}(a, \mu; a', \phi)$, we obtain

$$H(a_H, \mu_L + \delta; \phi) \;\geq\; \widetilde{H}(a_H, \mu_L + \delta; a'_L, \phi) \;>\; \widetilde{H}(a_H, \mu_L; a'_L, \phi) \;\geq\; H(a_H, \mu_L; \phi).$$

Hence $\mu_L \notin \mathcal{M}(a_H, \phi)$, i.e. every $\mu_H \in \mathcal{M}(a_H, \phi)$ must satisfy $\mu_H > \mu_L$. □

OA-D.4.2 Proof of Proposition 2

Proof of Proposition 2. Let $\mathcal{A}^1(\phi) = \{a \geq a_{\min} :\ \hat{y}(a, \phi) - \chi_f(\phi) \geq 0\}$ is the current-period feasibility set for undertaking R&D. We prove the extensive- and intensive-margin claims in turn.

Part (i): Extensive margin (threshold for I). Define the net gain from innovating:

$$\Delta(a, \phi) \;:=\; V^1(a, \phi) - V^0(a, \phi).$$

By Lemma 9, $V^1(a, \phi) = -\infty$ for $a \notin \mathcal{A}^1(\phi)$, hence $\Delta(a, \phi) = -\infty < 0$ there and $I^*(a, \phi) = 0$.

Now restrict attention to $a \in \mathcal{A}^1(\phi)$. By Lemma 9 (iii), $\Delta(\cdot, \phi)$ is continuous on $\mathcal{A}^1(\phi)$, and by Lemma 8, $\Delta(\cdot, \phi)$ is weakly increasing in a on $\mathcal{A}^1(\phi)$ (strictly increasing if $\epsilon > 0$).

Define the cutoff

$$\underline{a}(\phi) \;:=\; \inf\big\{a \in \mathcal{A}^1(\phi) :\ \Delta(a, \phi) \geq 0\big\},$$

with the convention $\inf \emptyset = \infty$. Monotonicity of $\Delta(\cdot, \phi)$ implies that, for $a \in \mathcal{A}^1(\phi)$,

$$\Delta(a, \phi) \geq 0 \quad\Longleftrightarrow\quad a \geq \underline{a}(\phi).$$

Thus $I^*(a, \phi) = 1$ if and only if $a \in \mathcal{A}^1(\phi)$ and $a \geq \underline{a}(\phi)$.

Part (ii): Intensive margin (monotonicity and strict monotonicity of μ^*). Fix $\phi \geq \underline{\phi}$ and consider the conditional problem given $I = 1$. Recall the μ-feasible set

$$\Gamma_\mu(a, \phi) \;:=\; \big\{\mu \geq 0 :\ y(a, \phi) - \chi_f(\phi) - \chi_v(\mu; \phi) \geq 0\big\},$$

and the post-savings value

$$H(a,\mu;\phi) \;:=\; \max_{a' \geq a_{\min}} \Big\{ U\big(y(a,\phi) - a' - \chi_v(\mu;\phi) - \chi_f(\phi)\big) + \beta\, \mathbb{E}[V(a',\phi') \mid \phi, 1, \mu] \Big\},$$

with infeasible choices (those implying $c < 0$ under the $U(0) = -\infty$ convention) assigned value $-\infty$ as in Lemma 10.

Step 1 (weak monotonicity). Lemma 10 shows that $H(\cdot,\cdot;\phi)$ has increasing differences in (a,μ) on its effective domain. Lemma 11 then implies that the optimizer correspondence $\mathcal{M}(\cdot,\phi)$ is nondecreasing in a in the strong set order. Since $\mathcal{M}(a,\phi)$ is nonempty and compact, the maximal selection $\mu^*(a,\phi) := \max \mathcal{M}(a,\phi)$ is well-defined and nondecreasing in a.

Step 2 (strict monotonicity when $\epsilon > 0$). Assume $\epsilon > 0$ and take any $a_H > a_L$ in the investment region (so the regime $I = 1$ is chosen and feasible at both wealth levels). Let (a'_L, μ_L) be a joint maximizer of the two-dimensional conditional problem at (a_L, ϕ); existence follows from Lemma 9 applied to $I = 1$.

We claim that this maximizer satisfies $\mu_L > 0$ and strictly positive consumption. Indeed, if $\mu_L = 0$, then the productivity transition under $(I,\mu) = (1,0)$ coincides with that under $(I,\mu) = (0,0)$ (Lemma 5), while the firm additionally pays the fixed cost $\chi_f(\phi) > 0$, so the regime $I = 1$ cannot be optimal—a contradiction. Hence $\mu_L > 0$. Moreover, under $\epsilon > 0$ with $\lim_{c \to 0^+} U(c) = +\infty$ (equivalently $c > 0$ in the admissible set), any optimizer must satisfy $c_L > 0$.

Therefore the hypotheses of Lemma 12 (iii) are met at (a_L, ϕ), which implies that for any $\mu_H \in \mathcal{M}(a_H, \phi)$, we have $\mu_H > \mu_L$. In particular, taking $\mu_H = \mu^*(a_H, \phi) = \max \mathcal{M}(a_H, \phi)$ and $\mu_L = \mu^*(a_L, \phi) = \max \mathcal{M}(a_L, \phi)$ yields

$$\mu^*(a_H, \phi) \;>\; \mu^*(a_L, \phi),$$

so $\mu^*(\cdot, \phi)$ is strictly increasing in a on the investment region. □

OA-D.5 Lending Policy Reforms

To analyze the impact of policies expending access to tangible/intangible collateral, we treat the collateral parameters θ (tangible collateralizability) and η (intangible collateralizability) as comparative static parameters. We first establish Lemmas 13 through 18 in Section OA-D.5.1. Subsequently, we provide the proof of Proposition 3 in Section OA-D.5.2.

OA-D.5.1 Lemmas for the Proof of Proposition 3

Lemma 13 (Monotonicity of Resources and Value in Parameters on Collateralizability)**.** The profit function $\pi(a, \phi; \theta, \eta)$ is non-decreasing in θ and η. Specifically:

(i) For unconstrained firms ($a \geq \hat{a}(\phi)$), the constraint is slack, so $\frac{\partial \pi}{\partial \theta} = \frac{\partial \pi}{\partial \eta} = 0$.

(ii) For constrained firms ($a < \hat{a}(\phi)$), profits are strictly increasing in financial development ($\frac{\partial \pi}{\partial \theta} > 0$ and $\frac{\partial \pi}{\partial \eta} > 0$) whenever the marginal product of capital exceeds the rental rate.

Consequently, the per-period resource function $y(a, \phi; \theta, \eta)$ is non-decreasing in θ and η. Furthermore, since the feasible set of the Bellman equation expands with θ and η while the objective function remains the same, the value function $V(a, \phi; \theta, \eta)$ is non-decreasing in θ and η (by the monotonicity of the Bellman operator).

Proof of Lemma 13. Fix (a, ϕ) with $a \geq a_{\min}$ and $\phi \geq \underline{\phi}$ and let the borrowing limit be

$$\bar{k}(a, \phi; \theta, \eta) := \theta a + \eta \Psi(\phi).$$

The static profit problem in (OA-12) can be written as

$$\pi(a, \phi; \theta, \eta) = \max_{k,l} \Big\{ R(\phi, k, l) - wl - (r + \delta)k \Big\} \quad \text{s.t.} \quad k \leq \bar{k}(a, \phi; \theta, \eta),$$

where $R(\phi, k, l) := \left(\phi^{\alpha_\phi} k^{\alpha_k} l^{\alpha_l}\right)^{\frac{\sigma-1}{\sigma}}$.

Step 1. Lagrangian and envelope derivative. Consider the Lagrangian

$$\mathcal{L}(k, l, \lambda; \theta, \eta) \;=\; R(\phi, k, l) - wl - (r + \delta)k + \lambda\big(\bar{k}(a, \phi; \theta, \eta) - k\big),$$

with multiplier $\lambda \geq 0$ on the borrowing constraint. The first-order condition w.r.t. k is

$$\frac{\partial R(\phi, k, l)}{\partial k} - (r + \delta) - \lambda \;=\; 0.$$

Evaluated at the optimum (k^*, l^*, λ^*), the marginal revenue product of capital is

$$M(a, \phi; w, \theta, \eta) \;:=\; \frac{\partial R(\phi, k^*, l^*)}{\partial k} \;=\; r + \delta + \lambda^*.$$

By the envelope theorem, the derivative of the maximized profit with respect to the borrowing limit $\bar{k}$ is

$$\frac{\partial \pi(a, \phi; \theta, \eta)}{\partial \bar{k}} \;=\; \lambda^*.$$

Using the chain rule with respect to a generic financial parameter $\tau \in \{\theta, \eta\}$, we obtain

$$\frac{\partial \pi(a, \phi; \theta, \eta)}{\partial \tau} = \frac{\partial \pi}{\partial \bar{k}} \frac{\partial \bar{k}}{\partial \tau} = \lambda^* \frac{\partial \bar{k}(a, \phi; \theta, \eta)}{\partial \tau}. \tag{OA-40}$$

Since

$$\frac{\partial \bar{k}}{\partial \theta} = a > 0, \qquad \frac{\partial \bar{k}}{\partial \eta} = \Psi(\phi) > 0,$$

it follows that the sign of $\partial \pi / \partial \tau$ is determined by λ^*.

Step 2. Constrained vs. unconstrained firms. By complementary slackness,

$$\lambda^* \big(\bar{k}(a, \phi; \theta, \eta) - k^*\big) = 0.$$

Hence

- If the firm is *unconstrained*, the borrowing constraint is slack, i.e. $k^* < \bar{k}(a, \phi; \theta, \eta)$, which implies $\lambda^* = 0$ and thus
$$\frac{\partial \pi}{\partial \theta} = \frac{\partial \pi}{\partial \eta} = 0.$$
This corresponds to the case $a \geq \hat{a}(\phi)$ in the text, where $M(a, \phi; \theta, \eta) = r + \delta$.

- If the firm is *constrained*, the constraint binds, i.e. $k^* = \bar{k}(a, \phi; \theta, \eta)$. In this case $\lambda^* > 0$ if and only if the marginal revenue product of capital exceeds the rental rate:
$$M(a, \phi; \theta, \eta) > r + \delta \quad \Longleftrightarrow \quad \lambda^* > 0.$$
Combining this with (OA-40) and the fact that $\partial \bar{k} / \partial \theta > 0$ and $\partial \bar{k} / \partial \eta > 0$, we obtain
$$\frac{\partial \pi(a, \phi; \theta, \eta)}{\partial \theta} > 0, \qquad \frac{\partial \pi(a, \phi; \theta, \eta)}{\partial \eta} > 0,$$
with strict inequality whenever $M(a, \phi; \theta, \eta) > r + \delta$ and $a \geq a_{\min}$ in the case of θ.

Thus, for each fixed (a, ϕ), $\pi(a, \phi; \theta, \eta)$ is non-decreasing in each of the financial parameters θ and η. This proves items (i)–(ii) in the lemma.

Step 3. Monotonicity of per-period resources. Per-period resources are given by

$$y(a, \phi; \theta, \eta) = \pi(a, \phi; \theta, \eta) + (1 + r)a.$$

The term $(1 + r)a$ does not depend on (θ, η), and $\pi(a, \phi; \theta, \eta)$ is non-decreasing in (θ, η) by Step 2. Therefore, $y(a, \phi; \theta, \eta)$ is non-decreasing in θ and in η.

Step 4. Monotonicity of the value function. Fix (θ_L, η_L) and (θ_H, η_H) with $\theta_H \geq \theta_L$ and $\eta_H \geq \eta_L$. Denote by V_L and V_H the corresponding value functions. For any state (a, ϕ), any admissible control (I, a', μ, c) that is feasible under (θ_L, η_L) satisfies the budget constraint

$$c + a' + \chi_v(\mu; \phi) + I \cdot \chi_f(\phi) \leq y(a, \phi; \theta_L, \eta_L).$$

Since $y(a, \phi; \theta_H, \eta_H) \geq y(a, \phi; \theta_L, \eta_L)$ by Step 3, the same control is also feasible under (θ_H, η_H), possibly with slack. Because U is strictly increasing in c, the maximal attainable value at (a, ϕ) under (θ_H, η_H) cannot be smaller than under (θ_L, η_L), given any continuation value function.

Formally, let $T_{\theta,\eta}$ denote the Bellman operator associated with parameters (θ, η). For any bounded W and any (a, ϕ),

$$(T_{\theta_H,\eta_H} W)(a, \phi) \;\geq\; (T_{\theta_L,\eta_L} W)(a, \phi),$$

since the maximization under (θ_H, η_H) is taken over a superset of the feasible choices under (θ_L, η_L) and the per-period utility is increasing in c. Thus, $T_{\theta_H,\eta_H} W \geq T_{\theta_L,\eta_L} W$ pointwise.

Starting from any common initial function W_0 and iterating,

$$T^n_{\theta_H,\eta_H} W_0 \;\geq\; T^n_{\theta_L,\eta_L} W_0 \quad \text{for all } n \geq 1.$$

By standard contraction mapping arguments (Lemma 4), $T^n_{\theta,\eta} W_0$ converges uniformly to $V_{\theta,\eta}$ as $n \to \infty$. Taking limits in n yields

$$V_H(a, \phi) = V(a, \phi; \theta_H, \eta_H) \;\geq\; V(a, \phi; \theta_L, \eta_L) = V_L(a, \phi)$$

for all (a, ϕ). Hence the value function $V(a, \phi; \theta, \eta)$ is non-decreasing in each of the financial development parameters θ and η. □

Lemma 14 (Increasing Differences in Current-Period Utility)**.** Fix the state (a, ϕ) and let $\tau \in \{\theta, \eta\}$ denote a financial development parameter. For clarity, write $y(\tau) := y(a, \phi; \tau)$.

The current-period utility from any given total of R&D expenditure plus savings

$$\mathcal{E} \;=\; \chi_v(\mu; \phi) + I \cdot \chi_f(\phi) + a',$$

defined by

$$\tilde{U}(\mathcal{E}, \tau) := U\big(y(\tau) - \mathcal{E}\big),$$

has *increasing differences* in $(\mathcal{E}, \tau)$.

More precisely, for any expenditure levels $\mathcal{E}' > \mathcal{E}$ and financial parameters $\tau_H > \tau_L$ such that $y(\tau_i) - \mathcal{E}' \geq 0$ for $i \in \{H, L\}$,

$$\tilde{U}(\mathcal{E}', \tau_H) - \tilde{U}(\mathcal{E}, \tau_H) \;\geq\; \tilde{U}(\mathcal{E}', \tau_L) - \tilde{U}(\mathcal{E}, \tau_L).$$

Proof of Lemma 14. Fix (a, ϕ) and let $\tau \in \{\theta, \eta\}$ be a financial development parameter. For notational simplicity, write $y(\tau) := y(a, \phi; \tau)$. By Lemma 13, $y(\tau)$ is non-decreasing in τ.

Let $\tau_H > \tau_L$ and set

$$y_H := y(\tau_H), \qquad y_L := y(\tau_L),$$

so that $y_H \geq y_L$. For financially constrained firms we in fact have $y_H > y_L$; for unconstrained firms $y_H = y_L$.

Fix expenditure levels $\mathcal{E}', \mathcal{E}$ such that

$$\mathcal{E}' > \mathcal{E} \geq a_{\min} \quad \text{and} \quad y_i - \mathcal{E}' \geq 0, \quad i \in \{H, L\},$$

so that all arguments of U below lie in the domain $[0, \infty)$.

Define, for each $i \in \{H, L\}$, the function

$$g_i(z) := U(y_i - z), \qquad z \in [\mathcal{E}, \mathcal{E}'].$$

Since $U \in C^2$ on $[0, \infty)$, each g_i is continuously differentiable on $[\mathcal{E}, \mathcal{E}']$, with derivative

$$g_i'(z) = \frac{\partial}{\partial z} U(y_i - z) = -U'(y_i - z).$$

By the Fundamental Theorem of Calculus,

$$U(y_i - \mathcal{E}') - U(y_i - \mathcal{E}) \;=\; g_i(\mathcal{E}') - g_i(\mathcal{E}) \;=\; \int_{\mathcal{E}}^{\mathcal{E}'} g_i'(z)\, dz \;=\; \int_{\mathcal{E}}^{\mathcal{E}'} -U'(y_i - z)\, dz. \tag{OA-41}$$

Now fix any $z \in [\mathcal{E}, \mathcal{E}']$ and compare the integrands for $i = H$ and $i = L$. Because $y_H \geq y_L$,

$$y_H - z \;\geq\; y_L - z.$$

Given the CRRA utility function, we have,

$$U'(y_H - z) \;\leq\; U'(y_L - z).$$

Multiplying by -1 reverses the inequality:

$$-U'(y_H - z) \ \geq \ -U'(y_L - z).$$

Integrating this pointwise inequality over $z \in [\mathcal{E}, \mathcal{E}']$ yields

$$\int_{\mathcal{E}}^{\mathcal{E}'} -U'(y_H - z)\, dz \ \geq \ \int_{\mathcal{E}}^{\mathcal{E}'} -U'(y_L - z)\, dz.$$

Using the representation (OA-41), we obtain

$$U(y_H - \mathcal{E}') - U(y_H - \mathcal{E}) \ \geq \ U(y_L - \mathcal{E}') - U(y_L - \mathcal{E}).$$

Recalling the definition $\tilde{U}(\mathcal{E}, \tau) := U\big(y(\tau) - \mathcal{E}\big)$, we have shown that for any $\mathcal{E}' > \mathcal{E}$ and any $\tau_H > \tau_L$,

$$\tilde{U}(\mathcal{E}', \tau_H) - \tilde{U}(\mathcal{E}, \tau_H) \ \geq \ \tilde{U}(\mathcal{E}', \tau_L) - \tilde{U}(\mathcal{E}, \tau_L).$$

□

Remark 4. Economically, this means that as financial development improves (higher τ and hence higher resources $y(\tau)$), the marginal utility cost of raising expenditure $\mathcal{E}$ (i.e. of diverting resources to R&D/savings) becomes smaller in absolute value, which is the desired wealth effect.

Lemma 15 (Productivity and Tangible-collateral Complementarity of y)**.** Fix $\eta = 0$ and let $\theta \geq 1$ denote the tangible collateral parameter. For each $(a, \phi) \in [a_{\min}, \infty) \times [\underline{\phi}, \infty)$, consider the per-period resource function

$$y(a, \phi; \theta) \ = \ \Gamma(w)\, \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \left[\frac{1-\tilde{\alpha}_l}{\tilde{\alpha}_k} M(a, \phi; \theta)^{-\frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} - (r+\delta)\, M(a, \phi; \theta)^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \right] + (1+r)a,$$

where, with $\eta = 0$,

$$M(a, \phi; \theta) = \max \left\{ r + \delta, \ \frac{\Gamma(w)^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}}\, \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_l}}}{(\theta a)^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}}} \right\}.$$

For each fixed $a \geq a_{\min}$:

(i) On the unconstrained region where the borrowing constraint does not bind (i.e. $M(a, \phi; \theta) = r + \delta$), $y(a, \phi; \theta)$ is independent of θ, so

$$\frac{\partial^2 y(a, \phi; \theta)}{\partial \phi\, \partial \theta} = 0.$$

(ii) On the constrained region where the borrowing constraint binds (i.e. $M(a,\phi;\theta)$ is given by the second term in the max), $y(a,\phi;\theta)$ is strictly supermodular in (ϕ,θ):

$$\frac{\partial^2 y(a,\phi;\theta)}{\partial\phi\,\partial\theta} \;>\; 0.$$

Consequently, for every $a \geq a_{\min}$, the function $y(a,\phi;\theta)$ has (weakly) increasing differences in (ϕ,θ), with strict increasing differences whenever the borrowing constraint binds.

Proof of Lemma 15. Set $\eta = 0$ throughout. For clarity, define

$$\varepsilon_1 := \frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_l} > 0, \qquad \varepsilon_2 := \frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l} > 0.$$

Unconstrained region. If the borrowing constraint does not bind at $(a,\phi;\theta)$, then

$$M(a,\phi;\theta) = r+\delta$$

is constant, independent of both ϕ and θ. Substituting this constant M into (OA-20) shows that $y(a,\phi;\theta)$ depends on ϕ but not on θ when the constraint is slack. Thus, for such $(a,\phi;\theta)$,

$$\frac{\partial y(a,\phi;\theta)}{\partial\theta} = 0 \quad\Rightarrow\quad \frac{\partial^2 y(a,\phi;\theta)}{\partial\phi\,\partial\theta} = 0,$$

establishing part (i).

Constrained region. Now consider the region where the borrowing constraint binds. With $\eta = 0$, the borrowing constraint is simply $k \leq \theta a$. When it binds, optimal capital satisfies $k^*(a,\phi;\theta) = \theta a$, and from (OA-16) we have

$$M(a,\phi;\theta) = \frac{\Gamma(w)^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}}\,\phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_l}}}{(\theta a)^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}}} = C_M\,\phi^{\varepsilon_1}\,(\theta a)^{-\varepsilon_2},$$

where

$$C_M := \Gamma(w)^{\frac{1-\tilde{\alpha}_k-\tilde{\alpha}_l}{1-\tilde{\alpha}_l}} > 0.$$

Substituting this expression for $M(a,\phi;\theta)$ into (OA-20) and collecting terms yields a closed-form representation of y on the constrained region. To keep notation compact, define the positive constants

$$A_1 := \frac{1-\tilde{\alpha}_l}{\tilde{\alpha}_k}, \qquad \delta_1 := \frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}, \qquad \delta_2 := \frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}.$$

Then, for binding M,

$$M^{-\delta_1} = C_M^{-\delta_1}\, \phi^{-\varepsilon_1\delta_1}\, (\theta a)^{\varepsilon_2\delta_1},$$

$$M^{-\delta_2} = C_M^{-\delta_2}\, \phi^{-\varepsilon_1\delta_2}\, (\theta a)^{\varepsilon_2\delta_2}.$$

Plugging these into (OA-20) and simplifying the exponents on ϕ gives

$$\begin{aligned} y(a, \phi; \theta) &= \Gamma(w)\, \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \Big[A_1 M^{-\delta_1} - (r+\delta) M^{-\delta_2}\Big] + (1+r)a \\ &= \Gamma(w) \Big[A_1 C_M^{-\delta_1}\, \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l} - \varepsilon_1\delta_1} (\theta a)^{\varepsilon_2\delta_1} - (r+\delta) C_M^{-\delta_2}\, \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l} - \varepsilon_1\delta_2} (\theta a)^{\varepsilon_2\delta_2}\Big] + (1+r)a. \end{aligned}$$

Using the definitions of $\varepsilon_1, \delta_1, \delta_2$ and straightforward algebra,

$$\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l} - \varepsilon_1\delta_1 = \varepsilon_1, \qquad \frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l} - \varepsilon_1\delta_2 = 0,$$

and

$$\varepsilon_2\delta_1 = \frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_l}, \qquad \varepsilon_2\delta_2 = 1.$$

Therefore we can rewrite $y(a, \phi; \theta)$ on the constrained region as

$$y(a, \phi; \theta) = B_1 \cdot (\theta a)^{\frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_l}}\, \phi^{\varepsilon_1} - B_2 \cdot (\theta a) + (1+r)a, \tag{OA-42}$$

where

$$B_1 := \Gamma(w)\, A_1\, C_M^{-\delta_1} > 0, \qquad B_2 := \Gamma(w)\, (r+\delta)\, C_M^{-\delta_2} > 0,$$

and all exponents are strictly positive by $1-\tilde{\alpha}_k-\tilde{\alpha}_l > 0$ and $1-\tilde{\alpha}_l > 0$.

Now compute the relevant derivatives. From (OA-42),

$$\frac{\partial y(a, \phi; \theta)}{\partial \phi} = B_1\, \varepsilon_1\, (\theta a)^{\frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_l}}\, \phi^{\varepsilon_1 - 1},$$

and differentiating again with respect to θ gives

$$\frac{\partial^2 y(a, \phi; \theta)}{\partial \theta\, \partial \phi} = B_1\, \varepsilon_1\, \frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_l}\, a \cdot (\theta a)^{\frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_l} - 1} \phi^{\varepsilon_1 - 1}.$$

Since $B_1 > 0$, $\varepsilon_1 > 0$, $\tilde{\alpha}_k > 0$, $1-\tilde{\alpha}_l > 0$, and $a \geq a_{\min}$, $\theta > 0$, $\phi \geq \underline{\phi}$, every factor in the last expression is strictly positive. Hence

$$\frac{\partial^2 y(a, \phi; \theta)}{\partial \theta\, \partial \phi} > 0 \quad \text{whenever the borrowing constraint binds,}$$

establishing part (ii).

Combining (i) and (ii) shows that for each $a \geq a_{\min}$ the function $y(a, \phi; \theta)$ has weakly increasing differences in (ϕ, θ), with strict increasing differences on the region where the constraint is binding. $\square$

Lemma 16 (Productivity and Intangible-collateral Complementarity of y)**.** Fix $\theta \geq 1$ and suppose Assumption 2 holds, i.e. $\upsilon_\phi > \sup_\phi \varsigma_\Psi(\phi)$. Let

$$y(a, \phi; \eta) \equiv y(a, \phi; \theta, \eta)$$

denote per-period resources, treating θ as fixed.

Then for each fixed $a \geq a_{\min}$:

(i) On the unconstrained region where the borrowing constraint does not bind, $y(a, \phi; \eta)$ is independent of η, so
$$\frac{\partial^2 y(a, \phi; \eta)}{\partial \phi \, \partial \eta} = 0.$$

(ii) On the constrained region where the borrowing constraint binds, $y(a, \phi; \eta)$ is strictly supermodular in (ϕ, η):
$$\frac{\partial^2 y(a, \phi; \eta)}{\partial \phi \, \partial \eta} \; > \; 0.$$

Consequently, for every $a \geq a_{\min}$, the function $y(a, \phi; \eta)$ has (weakly) increasing differences in (ϕ, η), with strict increasing differences whenever the borrowing constraint binds.

Proof of Lemma 16. Fix $\theta \geq 1$ and regard η as the financial development parameter. Write $y(a, \phi; \eta) \equiv y(a, \phi; \theta, \eta)$ and $\pi(a, \phi; \eta) \equiv \pi(a, \phi; \theta, \eta)$ for brevity. Throughout we fix $a \geq a_{\min}$ and vary (ϕ, η).

By definition,

$$y(a, \phi; \eta) = \pi(a, \phi; \eta) + (1 + r)a,$$

so

$$\frac{\partial^2 y(a, \phi; \eta)}{\partial \phi \, \partial \eta} = \frac{\partial^2 \pi(a, \phi; \eta)}{\partial \phi \, \partial \eta}.$$

Step 1: Unconstrained region. Recall from (OA-12) that profits are given by

$$\pi(a, \phi; \eta) = \max_{k, l \geq 0} \left\{ (\phi^{\alpha_\phi} k^{\alpha_k} l^{\alpha_l})^{\frac{\sigma-1}{\sigma}} - wl - (r + \delta)k \right\} \quad \text{s.t.} \quad k \leq \bar{k}(a, \phi; \eta),$$

where

$$\bar{k}(a, \phi; \eta) := \theta a + \eta \Psi(\phi).$$

If the borrowing constraint is slack at (a, ϕ, η), then the unconstrained optimum $(k^u(\phi), l^u(\phi))$ is independent of (a, θ, η). Equivalently, from (OA-16) and (OA-20), in the unconstrained region

$$M(a, \phi; w) = r + \delta$$

and $y(a, \phi; \eta)$ does not depend on η. Hence, for such (a, ϕ, η),

$$\frac{\partial y(a, \phi; \eta)}{\partial \eta} = 0 \quad \Rightarrow \quad \frac{\partial^2 y(a, \phi; \eta)}{\partial \phi \, \partial \eta} = 0,$$

which proves part (i) of the lemma.

Step 2: Constrained region. Now suppose the borrowing constraint binds at (a, ϕ, η), so that the desired unconstrained capital exceeds $\bar{k}(a, \phi; \eta)$ and the firm is capital constrained. In this case the optimal capital input is

$$k^*(a, \phi; \eta) = \bar{k}(a, \phi; \eta) = \theta a + \eta \Psi(\phi),$$

while labor l^* remains optimally chosen.

Define the "reduced" static profit function

$$\widetilde{\pi}(k, \phi) := \max_{l \geq 0} \left\{ (\phi^{\alpha_\phi} k^{\alpha_k} l^{\alpha_l})^{\frac{\sigma - 1}{\sigma}} - wl - (r + \delta) k \right\}.$$

Then, on the constrained region,

$$\pi(a, \phi; \eta) = \widetilde{\pi}\big(\bar{k}(a, \phi; \eta), \phi\big).$$

Since $y(a, \phi; \eta) = \pi(a, \phi; \eta) + (1 + r)a$, we have

$$\frac{\partial^2 y(a, \phi; \eta)}{\partial \phi \, \partial \eta} = \frac{\partial^2 \pi(a, \phi; \eta)}{\partial \phi \, \partial \eta} \quad \text{whenever the constraint binds.}$$

Let subscripts denote partial derivatives of $\widetilde{\pi}$, e.g. $\widetilde{\pi}_k := \partial \widetilde{\pi} / \partial k$. By envelope arguments, $\widetilde{\pi}_k(k, \phi)$ is the (net) marginal value of an additional unit of capital; in the constrained region firms would strictly prefer more capital, so

$$\widetilde{\pi}_k\big(\bar{k}(a, \phi; \eta), \phi\big) > 0 \quad \text{whenever } k^*(a, \phi; \eta) = \bar{k}(a, \phi; \eta) \text{ is binding.} \tag{OA-43}$$

For notational convenience, let

$$\bar{k} := \bar{k}(a, \phi; \eta) = \theta a + \eta \Psi(\phi),$$

Then

$$\bar{k}_a = \theta > 0, \qquad \bar{k}_\eta = \Psi(\phi) > 0, \qquad \bar{k}_\phi = \eta\Psi'(\phi) \geq 0, \qquad \bar{k}_{\eta\phi} = \Psi'(\phi) > 0, \qquad \bar{k}_{a\phi} = 0.$$

Step 3: Relating $\pi_{\phi\eta}$ to $\pi_{a\phi}$. By the chain rule, in the constrained region we have

$$\pi_a(a, \phi; \eta) = \widetilde{\pi}_k(\bar{k}, \phi)\, \bar{k}_a = \theta\, \widetilde{\pi}_k(\bar{k}, \phi), \tag{OA-44}$$

$$\pi_\eta(a, \phi; \eta) = \widetilde{\pi}_k(\bar{k}, \phi)\, \bar{k}_\eta = \widetilde{\pi}_k(\bar{k}, \phi)\, \Psi(\phi), \tag{OA-45}$$

and hence

$$\pi_{a\phi}(a, \phi; \eta) = \frac{\partial}{\partial \phi}\Big[\theta\, \widetilde{\pi}_k(\bar{k}, \phi)\Big] = \theta\big[\widetilde{\pi}_{k\phi}(\bar{k}, \phi) + \widetilde{\pi}_{kk}(\bar{k}, \phi)\, \bar{k}_\phi\big], \tag{OA-46}$$

$$\pi_{\phi\eta}(a, \phi; \eta) = \frac{\partial}{\partial \phi}\Big[\widetilde{\pi}_k(\bar{k}, \phi)\, \Psi(\phi)\Big] = \big[\widetilde{\pi}_{k\phi}(\bar{k}, \phi) + \widetilde{\pi}_{kk}(\bar{k}, \phi)\, \bar{k}_\phi\big]\Psi(\phi) + \widetilde{\pi}_k(\bar{k}, \phi)\, \Psi'(\phi). \tag{OA-47}$$

Combining (OA-46) and (OA-47) yields the identity

$$\pi_{\phi\eta}(a, \phi; \eta) = \frac{\Psi(\phi)}{\theta}\, \pi_{a\phi}(a, \phi; \eta) + \widetilde{\pi}_k(K, \phi)\, \Psi'(\phi). \tag{OA-48}$$

Step 4: Sign of $\pi_{a\phi}$ and $\pi_{\phi\eta}$. By Lemma 3, under Assumptions 2, the profit function $\pi(a, \phi)$ is strictly supermodular in (a, ϕ) on the constrained region; that is,

$$\pi_{a\phi}(a, \phi; \eta) > 0 \qquad \text{whenever the borrowing constraint binds.}$$

In addition, by (OA-43) and $\Psi'(\phi) > 0$,

$$\widetilde{\pi}_k(K, \phi)\, \Psi'(\phi) > 0 \qquad \text{whenever the borrowing constraint binds.}$$

Since $\Psi(\phi) > 0$ and $\theta > 0$, the decomposition (OA-48) implies

$$\pi_{\phi\eta}(a, \phi; \eta) > 0 \qquad \text{for all constrained } (a, \phi, \eta).$$

Recalling that $y(a, \phi; \eta) = \pi(a, \phi; \eta) + (1 + r)a$, this yields

$$\frac{\partial^2 y(a, \phi; \eta)}{\partial \phi\, \partial \eta} = \frac{\partial^2 \pi(a, \phi; \eta)}{\partial \phi\, \partial \eta} = \pi_{\phi\eta}(a, \phi; \eta) > 0 \qquad \text{whenever the constraint binds,}$$

establishing part (ii).

Step 5: Increasing differences in (ϕ, η). We have shown that, for each fixed $a \geq a_{\min}$,

- on the unconstrained region, $y(a, \phi; \eta)$ is independent of η, so $\partial^2 y(a, \phi; \eta)/(\partial\phi\,\partial\eta) = 0$;
- on the constrained region, $\partial^2 y(a, \phi; \eta)/(\partial\phi\,\partial\eta) > 0$.

By Lemma 2, $y(a, \phi; \eta)$ is continuous in (a, ϕ) (and the static problem is continuous in η as well). Hence, for each fixed $a \geq a_{\min}$, $y(a, \phi; \eta)$ has (weakly) increasing differences in (ϕ, η), with *strict* increasing differences whenever the borrowing constraint binds. This proves the lemma. $\square$

Lemma 17 (Value-Difference Monotonicity in Productivity across θ)**.** Fix $\epsilon = 0$ and $\eta = 0$. Let θ vary in some ordered subset $\Theta \subset [1, \infty)$. For any $a' \geq a_{\min}$ and any $\theta_H, \theta_L \in \Theta$ with $\theta_H > \theta_L$, define

$$\Delta_{a',\theta_H,\theta_L}(\phi') \;:=\; V(a', \phi'; \theta_H) - V(a', \phi'; \theta_L), \qquad \phi' \geq \underline{\phi}.$$

For any given current state (ϕ, I, μ) and bounded shock $|\xi'| \leq \bar{\xi}$, the one-step support

$$\mathcal{Z}(\phi, I, \mu) = \left[e^{\rho \ln \phi + \bar{\mu} + I\mu - \sigma_\xi \bar{\xi}},\ e^{\rho \ln \phi + \bar{\mu} + I\mu + \sigma_\xi \bar{\xi}}\right]$$

is a compact interval, and $\Delta_{a',\theta_H,\theta_L}(\cdot)$ is nonnegative and nondecreasing on $\mathcal{Z}(\phi, I, \mu)$.

Proof of Lemma 17. Fix $\eta = 0$ throughout and let $\theta \in [1, \infty)$. For each θ, denote by T_θ the Bellman operator associated with parameter θ, acting on the weighted space $\mathcal{C}_\omega$ from Lemma 4, and by $V(\cdot, \cdot; \theta)$ its unique fixed point.

The proof proceeds in five steps.

Step 1. A class of candidate value functions.

For each $\theta \in \Theta$ define $\mathcal{V}_\theta$ as the set of continuous functions $W(\cdot, \cdot; \theta) \in \mathcal{C}_\omega$ such that

(a) for every $a \geq a_{\min}$, the map $\phi \mapsto W(a, \phi; \theta)$ is (weakly) increasing;

(b) for every $a \geq a_{\min}$ and every $\theta_H > \theta_L$ in Θ, the *value difference*

$$\Delta_W(a, \phi; \theta_H, \theta_L) := W(a, \phi; \theta_H) - W(a, \phi; \theta_L)$$

is nonnegative and (weakly) increasing in ϕ.

Let

$$\mathcal{V} \;:=\; \bigcap_{\theta\in\Theta} \mathcal{V}_\theta$$

be the set of functions W on $[a_{\min},\infty) \times [\underline{\phi},\infty) \times \Theta$ satisfying (a)–(b) for all $\theta \in \Theta$. The zero function $W^0(a,\phi;\theta) \equiv 0$ belongs to $\mathcal{V}$. Our goal is to show that for every $\theta \in \Theta$ the Bellman operator T_θ maps $\mathcal{V}$ into itself and that its fixed point $V(\cdot,\cdot;\theta)$ lies in $\mathcal{V}$. From this it will follow that, for every $a' \geq a_{\min}$ and $\theta_H > \theta_L$, $\Delta_{a',\theta_H,\theta_L}(\cdot)$ is nonnegative and nondecreasing on $[\underline{\phi},\infty)$, and hence also on $\mathcal{Z}(\phi, I, \mu) \subseteq [\underline{\phi},\infty)$.

Step 2. Increasing differences of current-period utility in (ϕ,θ).

Fix (a,θ) and a feasible choice triple $(I,a',\mu) \in \{0,1\} \times [a_{\min},+\infty) \times \mathbb{R}_+$. Define total non-consumption expenditure as

$$\mathcal{E}(\phi, I, \mu) \;:=\; a' + \chi_v(\mu;\phi) + I \cdot \chi_f(\phi),$$

so that the budget constraint implies

$$c(a,\phi;\theta,I,a',\mu) \;=\; y(a,\phi;\theta) - \mathcal{E}(\phi,I,\mu).$$

By Lemma 2, $y(a,\phi;\theta)$ is strictly increasing in ϕ. By Lemma 15, for each $a \geq a_{\min}$ the per-period resource function $y(a,\phi;\theta)$ has (weakly) increasing differences in (ϕ,θ). In particular, on the binding region

$$\mathcal{D}_\theta := \big\{(a,\phi) : \partial_\theta y(a,\phi;\theta) > 0\big\},$$

we have

$$\partial_\phi y > 0, \quad \partial_\theta y > 0, \quad \partial^2_{\phi\theta} y > 0, \qquad \text{on } \mathcal{D}_\theta,$$

while on the unconstrained region $\partial_\theta y = \partial^2_{\phi\theta} y = 0$.

On the binding region $\mathcal{D}_\theta$, the cross-partial derivative of the current utility

$$U\big(c(a,\phi;\theta,I,a',\mu)\big) = U\big(y(a,\phi;\theta) - \mathcal{E}(\phi,I,\mu)\big)$$

with respect to (ϕ,θ) is

$$\begin{aligned}\frac{\partial^2}{\partial\phi\,\partial\theta} U\big(c(a,\phi;\theta,I,a',\mu)\big) &= U''(c)\,\big(\partial_\phi y - \partial_\phi \mathcal{E}\big)\,\partial_\theta y + U'(c)\,\partial^2_{\phi\theta} y \\ &= U''(c)\,\partial_\phi y\,\partial_\theta y + U'(c)\,\partial^2_{\phi\theta} y \;-\; U''(c)\,\partial_\phi \mathcal{E}\,\partial_\theta y,\end{aligned}$$

where $c = c(a, \phi; \theta, I, a', \mu)$. Under CRRA preferences with $\epsilon = 0$, we have $U'(c) = 1$ and $U''(c) = 0$. So

$$\frac{\partial^2}{\partial\phi\,\partial\theta} U\big(c(a, \phi; \theta, I, a', \mu)\big) = \partial^2_{\phi\theta} y \; > \; 0 \quad \text{on} \quad \mathcal{D}_\theta.$$

On the unconstrained region, $\partial_\theta y = \partial^2_{\phi\theta} y = 0$, so the cross-partial is equal to zero. Therefore, for any fixed feasible choice (I, a', μ) the current-period utility term exhibits (weakly) increasing differences in (ϕ, θ) on $[\underline{\phi}, +\infty) \times \Theta$.

Step 3. Increasing differences of the continuation value.

Let $W \in \mathcal{V}$ be arbitrary. For fixed (a', θ_H, θ_L) with $\theta_H > \theta_L$ define

$$\Delta_W(a', \phi') := W(a', \phi'; \theta_H) - W(a', \phi'; \theta_L), \qquad \phi' \geq \underline{\phi}.$$

By property (b) in the definition of $\mathcal{V}$, $\Delta_W(a', \cdot)$ is nonnegative and (weakly) increasing on $[\underline{\phi}, +\infty)$.

Fix a current state (a, ϕ) and feasible choice (I, a', μ). By Lemma 5, conditional on (ϕ, I, μ) the next-period productivity ϕ' has compact support $\mathcal{Z}(\phi, I, \mu)$, and by property (F2+) in that lemma, for any $\phi_2 > \phi_1$ the conditional law of ϕ' given (ϕ_2, I, μ) strictly FOSD-dominates that given (ϕ_1, I, μ). Therefore, for any nonnegative and nondecreasing function h,

$$\mathbb{E}\big[h(\phi') \mid \phi_2, I, \mu\big] \; \geq \; \mathbb{E}\big[h(\phi') \mid \phi_1, I, \mu\big].$$

Applying this with $h = \Delta_W(a', \cdot)$ yields, for every $\phi_2 > \phi_1$,

$$\begin{aligned}
&\mathbb{E}\big[W(a', \phi'; \theta_H) \mid \phi_2, I, \mu\big] - \mathbb{E}\big[W(a', \phi'; \theta_L) \mid \phi_2, I, \mu\big] \\
&\quad = \mathbb{E}\big[\Delta_W(a', \phi') \mid \phi_2, I, \mu\big] \; \geq \; \mathbb{E}\big[\Delta_W(a', \phi') \mid \phi_1, I, \mu\big] \\
&\quad = \mathbb{E}\big[W(a', \phi'; \theta_H) \mid \phi_1, I, \mu\big] - \mathbb{E}\big[W(a', \phi'; \theta_L) \mid \phi_1, I, \mu\big].
\end{aligned}$$

Thus, for each fixed choice (I, a', μ), the continuation-value term

$$(\phi, \theta) \longmapsto \beta\, \mathbb{E}\big[W(a', \phi'; \theta) \mid \phi, I, \mu\big]$$

has (weakly) increasing differences in (ϕ, θ) on $[\underline{\phi}, +\infty) \times \Theta$. In particular, for $\theta_H > \theta_L$ the difference across θ is nonnegative and nondecreasing in ϕ.

Step 4. The Bellman operator preserves $\mathcal{V}$.

For a given $W \in \mathcal{V}$ and parameter $\theta \in \Theta$, the Bellman operator can be written as

$$T_\theta W(a,\phi) = \sup_{(I,a',\mu)\in\Gamma(a,\phi;\theta)} \left\{ U\big(c(a,\phi;\theta,I,a',\mu)\big) + \beta\,\mathbb{E}\big[W(a',\phi';\theta) \mid \phi, I, \mu\big] \right\},$$

where

$$\Gamma(a,\phi;\theta) := \Big\{(I,a',\mu) \in \{0,1\} \times [a_{\min}, +\infty) \times \mathbb{R}_+ : c(a,\phi;\theta,I,a',\mu) \geq 0\Big\}$$

is the feasible set implied by the budget constraint and $c \geq 0$.

By Steps 2 and 3, for each fixed choice (I,a',μ) the inner objective

$$F_W(a,\phi;\theta,I,a',\mu) := U\big(c(a,\phi;\theta,I,a',\mu)\big) + \beta\,\mathbb{E}\big[W(a',\phi';\theta) \mid \phi, I, \mu\big]$$

has (weakly) increasing differences in (ϕ,θ) on $[\underline{\phi}, +\infty) \times \Theta$. Moreover, using Lemma 13, for any fixed (a,ϕ) the feasible set $\Gamma(a,\phi;\theta)$ is nonempty, compact, and *increasing* in θ under the strong set order: if $\theta_H > \theta_L$ and $(I,a',\mu) \in \Gamma(a,\phi;\theta_L)$, then $(I,a',\mu) \in \Gamma(a,\phi;\theta_H)$ because $y(a,\phi;\theta_H) \geq y(a,\phi;\theta_L)$.

Using the monotone comparative-statics theorem for maximization with parameterized constraints (see e.g., **?**, or **?**), we have, for each fixed a,

$$\phi \longmapsto T_\theta W(a,\phi)$$

is (weakly) increasing on $[\underline{\phi}, +\infty)$ for every $\theta \in \Theta$, and, for any $\theta_H > \theta_L$, the difference

$$\Delta_{TW}(a,\phi;\theta_H,\theta_L) := T_{\theta_H} W(a,\phi) - T_{\theta_L} W(a,\phi)$$

is nonnegative and (weakly) increasing in ϕ on $[\underline{\phi}, +\infty)$. Thus $T_\theta W \in \mathcal{V}$ for every θ, and so $T : \mathcal{V} \to \mathcal{V}$.

Step 5. Iteration and convergence.

Initialize with $W^0(a,\phi;\theta) \equiv 0$ for all (a,ϕ,θ). Then $W^0 \in \mathcal{V}$. Define iterates

$$W^{n+1}(\cdot,\cdot;\theta) := T_\theta W^n(\cdot,\cdot;\theta), \qquad n \geq 0,\ \theta \in \Theta.$$

By Step 4, $W^n \in \mathcal{V}$ for all n.

By Lemma 4, for each fixed $\theta \in \Theta$ the operator T_θ is a contraction on $\mathcal{C}_\omega$ and has a unique fixed point $V(\cdot,\cdot;\theta)$. Therefore $W^n(\cdot,\cdot;\theta) \to V(\cdot,\cdot;\theta)$ in the $\|\cdot\|_\omega$ norm as $n \to \infty$. Convergence in this weighted supremum norm implies pointwise convergence on $[a_{\min}, +\infty) \times [\underline{\phi}, +\infty)$.

For any fixed a, $\theta_H > \theta_L$, and $\phi_2 > \phi_1 \geq \underline{\phi}$, property (b) for W^n gives

$$0 \;\leq\; \big[W^n(a,\phi_2;\theta_H) - W^n(a,\phi_2;\theta_L)\big] - \big[W^n(a,\phi_1;\theta_H) - W^n(a,\phi_1;\theta_L)\big].$$

Passing to the limit as $n \to \infty$ and using pointwise convergence, we obtain

$$0 \;\leq\; \big[V(a,\phi_2;\theta_H) - V(a,\phi_2;\theta_L)\big] - \big[V(a,\phi_1;\theta_H) - V(a,\phi_1;\theta_L)\big].$$

That is, for every $a \geq a_{\min}$ and $\theta_H > \theta_L$, the function

$$\phi \longmapsto \Delta_{a,\theta_H,\theta_L}(\phi) = V(a,\phi;\theta_H) - V(a,\phi;\theta_L)$$

is nonnegative and (weakly) increasing on $[\underline{\phi}, +\infty)$. Since $\mathcal{Z}(\phi, I, \mu) \subseteq [\underline{\phi}, +\infty)$ by Lemma 5, $\Delta_{a',\theta_H,\theta_L}$ is nonnegative and nondecreasing on $\mathcal{Z}(\phi, I, \mu)$ for every $a' > 0$, as stated in the lemma. $\square$

Lemma 18 (Value-Difference Monotonicity in Productivity across η)**.** Fix $\epsilon = 0$ and $\theta \geq 1$. Consider two intangible–collateral regimes: a benchmark regime $\eta_L = 0$ and a reform regime $\eta_H > 0$. For $j \in \{L, H\}$, let $V_j(a,\phi) := V(a,\phi;\theta,\eta_j)$, and define

$$\Delta V(a,\phi) \;:=\; V_H(a,\phi) - V_L(a,\phi), \qquad (a,\phi) \in [a_{\min}, +\infty) \times [\underline{\phi}, +\infty).$$

For every a, and for any given current state (ϕ, I, μ) and bounded shock $|\xi'| \leq \bar{\xi}$, the one-step support

$$\mathcal{Z}(\phi, I, \mu) = \Big[e^{\rho \ln \phi + \bar{\mu} + I\mu - \sigma_\xi \bar{\xi}},\; e^{\rho \ln \phi + \bar{\mu} + I\mu + \sigma_\xi \bar{\xi}}\Big]$$

is a compact interval, and $\Delta V(a,\cdot)$ is nonnegative and nondecreasing on $\mathcal{Z}(\phi, I, \mu)$.

Proof of Lemma 18. Fix $\theta \geq 1$. Consider two intangible-collateral regimes: a benchmark regime $\eta_L = 0$ and a reform regime $\eta_H > 0$. For notational convenience write

$$V_j(a,\phi) := V(a,\phi;\theta,\eta_j), \qquad j \in \{L, H\},$$

and

$$\Delta V(a,\phi) := V_H(a,\phi) - V_L(a,\phi).$$

The structure of the proof parallels that of Lemma 17, with the tangible-collateral parameter θ replaced by the intangible-collateral parameter η.

Step 1. Candidate class of value functions.

Define $\mathcal{V}$ as the set of pairs of continuous functions $\{W(\cdot,\cdot;\eta_L), W(\cdot,\cdot;\eta_H)\} \subset \mathcal{C}^2_\omega$ such that, for every $a \geq a_{\min}$,

(a) $\phi \mapsto W(a,\phi;\eta_j)$ is (weakly) increasing on $[\underline{\phi}, +\infty)$ for $j \in \{L, H\}$;

(b) the difference

$$\Delta_W(a,\phi) := W(a,\phi;\eta_H) - W(a,\phi;\eta_L)$$

is nonnegative and (weakly) increasing in ϕ on $[\underline{\phi}, +\infty)$.

The zero pair $W^0(a,\phi;\eta_j) \equiv 0$ belongs to $\mathcal{V}$.

For each fixed $\eta \in \{\eta_L, \eta_H\}$, let $T_{\theta,\eta}$ denote the Bellman operator associated with the parameter pair (θ, η) and $V(\cdot,\cdot;\theta,\eta)$ its unique fixed point (Lemma 4). We will show that, if $\{W(\cdot,\cdot;\eta_j)\}_{j=L,H} \in \mathcal{V}$, then the pair

$$\{T_{\eta_L} W(\cdot,\cdot;\eta_L),\, T_{\eta_H} W(\cdot,\cdot;\eta_H)\}$$

also lies in $\mathcal{V}$. Iteration from W^0 and contraction of each T_{η_j} then imply that the limit pair $\{V(\cdot,\cdot;\eta_L), V(\cdot,\cdot;\eta_H)\}$ belongs to $\mathcal{V}$, which is precisely the conclusion of the lemma.

Step 2. Increasing differences of current-period utility in (ϕ, η).

By Assumption 2 and Lemma 16, for each fixed $a \geq a_{\min}$ the resource function $y(a,\phi;\theta,\eta)$ has (weakly) increasing differences in (ϕ,η):

$$\partial_\phi y > 0, \quad \partial_\eta y > 0, \quad \partial^2_{\phi\eta} y > 0, \qquad \text{on } \mathcal{D}_\eta := \big\{(a,\phi) : \partial_\eta y(a,\phi;\theta) > 0\big\},$$

and $\partial_\eta y = \partial^2_{\phi\eta} y = 0$ on the unconstrained region.

As in Lemma 17, write the budget constraint in the form

$$c(a,\phi;\eta,I,a',\mu) = y(a,\phi;\theta,\eta) - \mathcal{E}(\phi,I,\mu), \qquad \mathcal{E}(\phi,I,\mu) := a' + \chi_v(\mu;\phi) + I \cdot \chi_f(\phi).$$

For fixed $(a,\eta) \in [a_{\min},\infty) \times \{\eta_L, \eta_H\}$ and a feasible choice triple (I, a', μ), the cross-partial derivative of $U(c(a,\phi;\eta,I,a',\mu))$ with respect to (ϕ,η) is

$$\frac{\partial^2}{\partial\phi\,\partial\eta} U\big(c(a,\phi;\eta,I,a',\mu)\big) = U''(c)\,\big(\partial_\phi y - \partial_\phi z\big)\,\partial_\eta y + U'(c)\,\partial^2_{\phi\eta} y,$$

where $c = c(a, \phi; \eta, I, a', \mu)$. Under CRRA preferences with $\epsilon = 0$,

$$\frac{\partial^2}{\partial\phi\,\partial\eta} U\big(c(a, \phi; \eta, I, a', \mu)\big) = \partial^2_{\phi\eta} y > 0 \qquad \text{on } \mathcal{D}_\eta.$$

On the unconstrained region, $\partial_\eta y = \partial^2_{\phi\eta} y = 0$, so the cross-partial is zero.

Therefore, for any fixed feasible choice (I, a', μ) the current-period utility term

$$(\phi, \eta) \longmapsto U\big(c(a, \phi; \eta, I, a', \mu)\big)$$

has (weakly) increasing differences in (ϕ, η) on $[\underline{\phi}, +\infty) \times \{\eta_L, \eta_H\}$.

Step 3. Increasing differences of the continuation value.

Let $\{W(\cdot, \cdot; \eta_j)\}_{j=L,H} \in \mathcal{V}$. For fixed a' define

$$\Delta_W(a', \phi') := W(a', \phi'; \eta_H) - W(a', \phi'; \eta_L), \qquad \phi' \geq \underline{\phi}.$$

By property (b) of $\mathcal{V}$, $\Delta_W(a', \cdot)$ is nonnegative and nondecreasing.

Fix a current state (a, ϕ) and feasible choice (I, a', μ). By Lemma 5, the conditional support of ϕ' given (ϕ, I, μ) is a compact interval $\mathcal{Z}(\phi, I, \mu)$, and by property (F2+) of that lemma, for any $\phi_2 > \phi_1$ the law of ϕ' given (ϕ_2, I, μ) strictly FOSD-dominates that given (ϕ_1, I, μ). Therefore,

$$\mathbb{E}\big[\Delta_W(a', \phi') \mid \phi_2, I, \mu\big] \;\geq\; \mathbb{E}\big[\Delta_W(a', \phi') \mid \phi_1, I, \mu\big] \quad \text{whenever } \phi_2 > \phi_1.$$

Since the transition law of ϕ' does not depend on η, the continuation-value term

$$(\phi, \eta) \longmapsto \beta\,\mathbb{E}\big[W(a', \phi'; \eta) \mid \phi, I, \mu\big]$$

inherits (weakly) increasing differences in (ϕ, η): for each ϕ and (I, a', μ) the difference across η_H and η_L is nonnegative and nondecreasing in ϕ.

Step 4. Preservation of $\mathcal{V}$ by the Bellman operators.

For each $\eta \in \{\eta_L, \eta_H\}$ the Bellman operator is

$$T_\eta W(a, \phi) = \sup_{(I, a', \mu) \in \Gamma(a, \phi; \theta, \eta)} \Big\{ U\big(c(a, \phi; \eta, I, a', \mu)\big) + \beta\,\mathbb{E}\big[W(a', \phi'; \eta) \mid \phi, I, \mu\big] \Big\},$$

with feasible set

$$\Gamma(a, \phi; \theta, \eta) := \Big\{ (I, a', \mu) \in \{0, 1\} \times [a_{\min}, +\infty) \times \mathbb{R}_+ : c(a, \phi; \eta, I, a', \mu) \geq 0 \Big\}.$$

By Lemma 13, for fixed (a, ϕ) the feasible set $\Gamma(a, \phi; \theta, \eta)$ is nonempty and expands with η in the strong set order; in particular, $\Gamma(a, \phi; \theta, \eta_L) \subseteq \Gamma(a, \phi; \theta, \eta_H)$.

By Steps 2 and 3, for each fixed (I, a', μ) the flow payoff

$$F_W(a, \phi; \eta, I, a', \mu) := U\big(c(a, \phi; \eta, I, a', \mu)\big) + \beta\, \mathbb{E}\big[W(a', \phi'; \eta) \mid \phi, I, \mu\big]$$

has (weakly) increasing differences in (ϕ, η).

Using the monotone comparative-statics theorem with parameterized constraints (see e.g., **?**, or **?**), we have, for every $a \geq a_{\min}$, $\phi \mapsto T_\eta W(a, \phi)$ is (weakly) increasing on $[\underline{\phi}, +\infty)$ for each $\eta \in \{\eta_L, \eta_H\}$, and that the difference

$$\Delta_{TW}(a, \phi) := T_{\eta_H} W(a, \phi) - T_{\eta_L} W(a, \phi)$$

is nonnegative and (weakly) increasing in ϕ. Hence the updated pair $\{T_{\eta_L} W(\cdot, \cdot; \eta_L), T_{\eta_H} W(\cdot, \cdot; \eta_H)\}$ again belongs to $\mathcal{V}$.

Step 5. Iteration and convergence.

Initialize with $W^0(a, \phi; \eta_j) \equiv 0$ for $j \in \{L, H\}$. Clearly $\{W^0(\cdot, \cdot; \eta_L), W^0(\cdot, \cdot; \eta_H)\} \in \mathcal{V}$. Define iterates

$$W^{n+1}(\cdot, \cdot; \eta_j) := T_{\eta_j} W^n(\cdot, \cdot; \eta_j), \qquad n \geq 0,\ j \in \{L, H\}.$$

By Step 4, the pair of iterates belongs to $\mathcal{V}$ for all n.

By Lemma 4, for each fixed η_j the operator T_{η_j} is a contraction on $\mathcal{C}_\omega$ and has a unique fixed point $V(\cdot, \cdot; \theta, \eta_j)$. Hence $W^n(\cdot, \cdot; \eta_j) \to V(\cdot, \cdot; \theta, \eta_j)$ in $\| \cdot \|_\omega$ as $n \to \infty$, and therefore pointwise on $[a_{\min}, +\infty) \times [\underline{\phi}, +\infty)$.

For any fixed $a \geq a_{\min}$ and $\phi_2 > \phi_1 \geq \underline{\phi}$, property (b) for W^n implies

$$0 \ \leq\ \big[W^n(a, \phi_2; \eta_H) - W^n(a, \phi_2; \eta_L)\big] - \big[W^n(a, \phi_1; \eta_H) - W^n(a, \phi_1; \eta_L)\big].$$

Taking limits as $n \to \infty$ and using pointwise convergence yields

$$0 \ \leq\ \big[V_H(a, \phi_2) - V_L(a, \phi_2)\big] - \big[V_H(a, \phi_1) - V_L(a, \phi_1)\big].$$

Thus, for every $a > 0$ the function

$$\phi \longmapsto \Delta V(a, \phi) = V_H(a, \phi) - V_L(a, \phi)$$

is nonnegative and (weakly) increasing on $[\underline{\phi}, +\infty)$. Since $\mathcal{Z}(\phi, I, \mu) \subseteq [\underline{\phi}, +\infty)$ by Lemma 5, $\Delta V(a, \cdot)$ is nonnegative and nondecreasing on $\mathcal{Z}(\phi, I, \mu)$, as stated. $\square$

OA-D.5.2 Proof of Proposition 3

We separate the proof into two parts: (1) the impact of θ on R&D, and (2) the impact of η on R&D. Throughout the proof, we fix $\epsilon = 0$.

The Impact of θ on R&D — The First Part of Proposition 3

Proof of the Impact of θ on R&D Target. Throughout the proof we fix $\eta = 0$ and treat θ as the only varying collateralizability parameter. For each θ let $V(\cdot, \cdot; \theta)$ denote the unique value function in the weighted space $\mathcal{C}_\omega$ (Lemma 4), and let $V^I(\cdot, \cdot; \theta)$, $I \in \{0, 1\}$, be the regime–specific value functions defined as in Lemma 9. We also write $y(a, \phi; \theta)$ for per-period resources in order to emphasise the dependence on θ. Fix $1 \leq \theta_L < \theta_H$ and $\phi \geq \underline{\phi}$.

For each (a, ϕ, θ) define the net value of innovation

$$\Delta(a, \phi; \theta) := V^1(a, \phi; \theta) - V^0(a, \phi; \theta).$$

By Lemma 8, for every (ϕ, θ) the function $a \mapsto \Delta(a, \phi; \theta)$ is increasing on the region where $I = 1$ is feasible, and Proposition 2 implies that, for each (ϕ, θ), there exists a threshold $\underline{a}(\phi; \theta)$ such that $I^*(a, \phi; \theta) = 1$ if and only if $a \geq \underline{a}(\phi; \theta)$.

Step 1: Increasing differences of the flow objective in (I, θ).

Fix a current state (a, ϕ) and a continuation choice vector $x = (a', \mu) \in [a_{\min}, +\infty) \times \mathbb{R}_+$. For $I \in \{0, 1\}$ and $\theta \in \{\theta_L, \theta_H\}$ define the one-period flow objective

$$\mathcal{O}(a, I; x, \phi; \theta) := U\big(y(a, \phi; \theta) - \mathcal{E}(I, \mu, a'; \phi)\big) + \beta\, \mathbb{E}\Big[V(a', \phi'; \theta) \mid \phi, I, \mu\Big],$$

where $\mathcal{E}(I, \mu, a'; \phi) := a' + \chi_v(\mu; \phi) + I \cdot \chi_f(\phi)$ is total expenditure on savings and R&D.

We claim that, for each fixed (a, ϕ, x), the map $(I, \theta) \mapsto \mathcal{O}(a, I; x, \phi; \theta)$ has (weakly) increasing differences on $\{0, 1\} \times \{\theta_L, \theta_H\}$.

(a) Current-period utility. Let $y(\theta) := y(a, \phi; \theta)$ and $x_0 := a' + \chi_v(\mu; \phi)$, $x_1 := x_0 + \chi_f(\phi)$. The current-period utility term equals $U(y(\theta) - x_1)$ under $I = 1$ and $U(y(\theta) - x_0)$ under $I = 0$. Lemma 14 (with $\chi \equiv \theta$) states that $(x, \theta) \mapsto U(y(\theta) - x)$ has increasing differences. Hence, for any $\theta_H > \theta_L$,

$$\big[U(y(\theta_H) - x_1) - U(y(\theta_H) - x_0)\big] - \big[U(y(\theta_L) - x_1) - U(y(\theta_L) - x_0)\big] \;\geq\; 0.$$

Thus the current-period utility component exhibits (weakly) increasing differences in (I, θ).

(b) Continuation value. For fixed (a', ϕ, μ) define

$$\mathcal{V}_c^I(a', \phi; \mu, \theta) := \mathbb{E}\big[V(a', \phi'; \theta) \mid \phi, I, \mu\big], \qquad I \in \{0, 1\}.$$

For $\theta_H > \theta_L$ let

$$\Delta_{a',\theta_H,\theta_L}(\phi') := V(a', \phi'; \theta_H) - V(a', \phi'; \theta_L), \qquad \phi' \geq \underline{\phi}.$$

By Lemma 13, $V(a', \phi'; \theta)$ is nondecreasing in θ, hence $\Delta_{a',\theta_H,\theta_L}(\phi') \geq 0$ for all ϕ'. Moreover, given $\epsilon = 0$, Lemma 17 implies that, for each fixed a', the function $\phi' \mapsto \Delta_{a',\theta_H,\theta_L}(\phi')$ is nonnegative and nondecreasing on $\mathcal{Z}(\phi, I, \mu)$.

Lemma 5, property (F1+), states that for any (ϕ, μ) with $\mu > 0$, the distribution of ϕ' under $(I = 1, \mu)$ strictly FOSD-dominates that under $(I = 0, 0)$; for $\mu = 0$ the two distributions coincide. Since $\Delta_{a',\theta_H,\theta_L}$ is nondecreasing, the standard characterization of first-order stochastic dominance therefore yields

$$\mathbb{E}\big[\Delta_{a',\theta_H,\theta_L}(\phi') \mid \phi, 1, \mu\big] \;\geq\; \mathbb{E}\big[\Delta_{a',\theta_H,\theta_L}(\phi') \mid \phi, 0, 0\big].$$

In terms of $\mathcal{V}_c^I$, this inequality is equivalent to

$$\big[\mathcal{V}_c^1(a', \phi; \mu, \theta_H) - \mathcal{V}_c^1(a', \phi; \mu, \theta_L)\big] - \big[\mathcal{V}_c^0(a', \phi; \mu, \theta_H) - \mathcal{V}_c^0(a', \phi; \mu, \theta_L)\big] \;\geq\; 0,$$

so the continuation-value component of $\mathcal{O}$ has (weakly) increasing differences in (I, θ).

Combining (a) and (b) we conclude that, for every fixed (a, ϕ, x), the function

$$(I, \theta) \longmapsto \mathcal{O}(a, I; x, \phi; \theta)$$

has (weakly) increasing differences in (I, θ) on $\{0, 1\} \times \{\theta_L, \theta_H\}$.

Step 2: From the flow objective to the net value of innovation.

To avoid dependence of the feasible set on (I, θ), extend $\mathcal{O}$ to all $x = (a', \mu) \in [a_{\min}, +\infty) \times \mathbb{R}_+$ by setting

$$\mathcal{O}(a, I; x, \phi; \theta) := -\infty \qquad \text{whenever} \qquad y(a, \phi; \theta) - \mathcal{E}(I, \mu, a'; \phi) < 0.$$

Under this convention Lemma 9 implies that, for each (a, ϕ, I, θ),

$$V^I(a, \phi; \theta) = \sup_{x \in [a_{\min}, +\infty) \times \mathbb{R}_+} \mathcal{O}(a, I; x, \phi; \theta),$$

and that the supremum is in fact attained.

Since $\mathcal{O}(a, I; x, \phi; \theta)$ has increasing differences in (I, θ) pointwise in x and the feasible set does not depend on (I, θ), Milgrom and Shannon's (**?**) maximization theorem implies that $(I, \theta) \mapsto V^I(a, \phi; \theta)$ inherits (weakly) increasing differences. Hence, for each fixed (a, ϕ) the function

$$\theta \longmapsto \Delta(a, \phi; \theta) = V^1(a, \phi; \theta) - V^0(a, \phi; \theta)$$

is nondecreasing on $[\theta_L, \theta_H]$.

Step 3: Extensive-margin comparative statics.

Fix $\phi \geq \underline{\phi}$. From Step 2 we have, for every $a \geq a_{\min}$,

$$\Delta(a, \phi; \theta_H) \;\geq\; \Delta(a, \phi; \theta_L).$$

Therefore

$$\big\{a \geq a_{\min} : \Delta(a, \phi; \theta_L) \geq 0\big\} \;\subseteq\; \big\{a \geq a_{\min} : \Delta(a, \phi; \theta_H) \geq 0\big\}.$$

By Lemma 8, for each (ϕ, θ) the function $a \mapsto \Delta(a, \phi; \theta)$ is increasing on the set where $I = 1$ is feasible. Together with the threshold defined in Proposition 2, this inclusion of sets implies

$$\underline{a}(\phi; \theta_H) \;\leq\; \underline{a}(\phi; \theta_L) \qquad \text{for all } \phi \geq \underline{\phi},$$

which proves part (i).

Step 4: Intensive-margin comparative statics.

Now fix a state (a, ϕ) such that

$$I^*(a, \phi; \theta_L) = I^*(a, \phi; \theta_H) = 1.$$

For each $\theta \in \{\theta_L, \theta_H\}$ define the conditional value of choosing R&D intensity $\mu \geq 0$ (given $I = 1$) by

$$H(a, \mu; \phi, \theta) := \sup_{a' \geq a_{\min}} \Big\{U\big(y(a, \phi; \theta) - a' - \chi_v(\mu; \phi) - \chi_f(\phi)\big) + \beta\, \mathbb{E}\big[V(a', \phi'; \theta) \mid \phi, 1, \mu\big]\Big\}.$$

By continuity of the objective, compactness of the feasible set in a' induced by the budget constraint $y(a, \phi; \theta) - a' - \chi_v(\mu; \phi) - \chi_f(\phi) \geq 0$, the supremum is attained. For each (a, ϕ, θ)

the argmax correspondence

$$\mathcal{M}(a,\phi;\theta) := \arg\max_{\mu\geq 0} H(a,\mu;\phi,\theta)$$

is therefore nonempty and compact. On the interior of the feasible region, strict concavity in μ implies that $\mathcal{M}(a,\phi;\theta)$ is single-valued; in that case we write $\mu^*(a,\phi;\theta)$ for the unique maximizer.[56]

We claim that, for each fixed (a,ϕ), the function $H(a,\mu;\phi,\theta)$ has (weakly) increasing differences in (μ,θ) on $[0,\infty)\times\{\theta_L,\theta_H\}$.

(a) Current-period utility. Fix (a',ϕ) and define

$$\mathcal{E}(\mu) := a' + \chi_v(\mu;\phi) + \chi_f(\phi), \qquad y(\theta) := y(a,\phi;\theta).$$

Since $\chi_v(\mu;\phi)$ is strictly increasing in μ, $\mathcal{E}(\mu)$ is strictly increasing. Lemma 14 (with $\tau \equiv \theta$) implies that $(\mathcal{E},\theta) \mapsto U(y(\theta)-x)$ has increasing differences. By the standard composition result for increasing differences (see e.g., **?**), it follows that

$$(\mu,\theta) \longmapsto U\big(y(\theta)-\mathcal{E}(\mu)\big)$$

has increasing differences in (μ,θ).

(b) Continuation value. For fixed (a',ϕ) define

$$\mathcal{V}_c^1(a',\mu;\phi,\theta) := \mathbb{E}\big[V(a',\phi';\theta)\mid \phi,1,\mu\big].$$

Let $\Delta_{a',\theta_H,\theta_L}$ be as in Step 1(b). Then

$$\mathcal{V}_c^1(a',\mu;\phi,\theta_H) - \mathcal{V}_c^1(a',\mu;\phi,\theta_L) = \mathbb{E}\big[\Delta_{a',\theta_H,\theta_L}(\phi')\mid \phi,1,\mu\big].$$

By Lemma 17, $\Delta_{a',\theta_H,\theta_L}$ is nonnegative and nondecreasing on $\mathcal{Z}(\phi,I,\mu)$ under $\epsilon=0$. Lemma 5, property (Fμ+), states that, conditional on $(\phi,I=1)$, the distribution of ϕ' is strictly FOSD- and MLR-increasing in μ. Hence, for any $\mu_H>\mu_L\geq 0$,

$$\mathbb{E}\big[\Delta_{a',\theta_H,\theta_L}(\phi')\mid \phi,1,\mu_H\big] \;\geq\; \mathbb{E}\big[\Delta_{a',\theta_H,\theta_L}(\phi')\mid \phi,1,\mu_L\big],$$

with strict inequality whenever the firm is financially constrained with positive probability in some future state. Equivalently,

$$\big[\mathcal{V}_c^1(a',\mu_H;\phi,\theta_H) - \mathcal{V}_c^1(a',\mu_L;\phi,\theta_H)\big] - \big[\mathcal{V}_c^1(a',\mu_H;\phi,\theta_L) - \mathcal{V}_c^1(a',\mu_L;\phi,\theta_L)\big] \;\geq\; 0,$$

[56] The strict concavity in μ arises from the strict convexity of R&D costs.

so the continuation-value component exhibits (weakly) increasing differences in (μ, θ).

Combining (a) and (b), we conclude that $H(a, \mu; \phi, \theta)$ has (weakly) increasing differences in (μ, θ) for each fixed (a, ϕ).

By Milgrom and Shannon's (**?**) monotone selection theorem, the argmax correspondence $\theta \mapsto \mathcal{M}(a, \phi; \theta)$ is nondecreasing in θ in the strong set order:

$$\min \mathcal{M}(a, \phi; \theta_H) \ \geq \ \min \mathcal{M}(a, \phi; \theta_L), \qquad \max \mathcal{M}(a, \phi; \theta_H) \ \geq \ \max \mathcal{M}(a, \phi; \theta_L).$$

On the interior, where $\mathcal{M}(a, \phi; \theta)$ is single-valued, this implies

$$\mu^*(a, \phi; \theta_H) \ \geq \ \mu^*(a, \phi; \theta_L)$$

whenever $I^*(a, \phi; \theta_L) = I^*(a, \phi; \theta_H) = 1$. This proves part (ii) and completes the proof of the proposition. $\square$

The Impact of η on R&D — The Second Part of Proposition 3

Proof of the Impact of η on R&D. Throughout the proof we fix $\theta \geq 1$ and treat η as the only varying collateralizability parameter. For each η let $V(\cdot, \cdot; \eta)$ denote the unique value function in the weighted space $\mathcal{C}_\omega$ (Lemma 4), and let $V^I(\cdot, \cdot; \eta)$, $I \in \{0, 1\}$, be the regime–specific value functions defined as in Lemma 9. We also write $y(a, \phi; \eta)$ for per-period resources to emphasize the dependence on η (holding θ fixed). Fix $0 = \eta_L < \eta_H$ and $\phi \geq \underline{\phi}$.

For each (a, ϕ, η) define the net value of innovation

$$\Delta(a, \phi; \eta) := V^1(a, \phi; \eta) - V^0(a, \phi; \eta).$$

By Lemma 8, for every (ϕ, η) the function $a \mapsto \Delta(a, \phi; \eta)$ is increasing on the region where $I = 1$ is feasible, and Proposition 2 implies that for each (ϕ, η) there exists a threshold $\underline{a}(\phi; \eta)$ such that $I^*(a, \phi; \eta) = 1$ if and only if $a \geq \underline{a}(\phi; \eta)$.

Step 1: Increasing differences of the flow objective in (I, η).

Fix a current state (a, ϕ) and a continuation choice vector $x = (a', \mu) \in [a_{\min}, +\infty) \times \mathbb{R}_+$. For $I \in \{0, 1\}$ and $\eta \in \{\eta_L, \eta_H\}$ define the one-period flow objective

$$\mathcal{O}(a, I; x, \phi; \eta) := U\big(y(a, \phi; \eta) - \mathcal{E}(I, \mu, a'; \phi)\big) + \beta \, \mathbb{E}\Big[V(a', \phi'; \eta) \mid \phi, I, \mu\Big],$$

where $\mathcal{E}(I, \mu, a'; \phi) := a' + \chi_v(\mu; \phi) + I \cdot \chi_f(\phi)$ is total expenditure on savings and R&D.

We claim that, for each fixed (a, ϕ, x), the map $(I, \eta) \mapsto \mathcal{O}(a, I; x, \phi; \eta)$ has (weakly) increasing differences on $\{0, 1\} \times \{\eta_L, \eta_H\}$.

(a) Current-period utility. Let $y(\eta) := y(a, \phi; \eta)$ and $x_0 := a' + \chi_v(\mu; \phi)$, $x_1 := x_0 + \chi_f(\phi)$. The current-period utility term equals $U(y(\eta) - x_1)$ under $I = 1$ and $U(y(\eta) - x_0)$ under $I = 0$. Lemma 14 (with $\tau \equiv \eta$) states that $(x, \eta) \mapsto U(y(\eta) - x)$ has increasing differences. Hence, for any $\eta_H > \eta_L$,

$$\big[U(y(\eta_H) - x_1) - U(y(\eta_H) - x_0)\big] - \big[U(y(\eta_L) - x_1) - U(y(\eta_L) - x_0)\big] \;\geq\; 0.$$

Thus the current-period utility component exhibits (weakly) increasing differences in (I, η).

(b) Continuation value. For fixed (a', ϕ, μ) define

$$\mathcal{V}_c^I(a', \phi; \mu, \eta) := \mathbb{E}\big[V(a', \phi'; \eta) \mid \phi, I, \mu\big], \qquad I \in \{0, 1\}.$$

For $\eta_H > \eta_L$ let

$$\Delta_{a', \eta_H, \eta_L}(\phi') := V(a', \phi'; \eta_H) - V(a', \phi'; \eta_L), \qquad \phi' \geq \underline{\phi}.$$

By Lemma 13, $V(a', \phi'; \eta)$ is nondecreasing in η, hence $\Delta_{a', \eta_H, \eta_L}(\phi') \geq 0$ for all ϕ'. Moreover, given $\epsilon = 0$, Lemma 18 implies that, for each fixed a', the function $\phi' \mapsto \Delta_{a', \eta_H, \eta_L}(\phi')$ is nonnegative and nondecreasing on $\mathcal{Z}(\phi, I, \mu)$.

Lemma 5, property (F1+), states that for any (ϕ, μ) with $\mu > 0$, the distribution of ϕ' under $(I = 1, \mu)$ strictly FOSD-dominates that under $(I = 0, 0)$; for $\mu = 0$ the two distributions coincide. Since $\Delta_{a', \eta_H, \eta_L}$ is nondecreasing, first-order stochastic dominance implies

$$\mathbb{E}\big[\Delta_{a', \eta_H, \eta_L}(\phi') \mid \phi, 1, \mu\big] \;\geq\; \mathbb{E}\big[\Delta_{a', \eta_H, \eta_L}(\phi') \mid \phi, 0, 0\big].$$

In terms of $\mathcal{V}_c^I$, this inequality is equivalent to

$$\big[\mathcal{V}_c^1(a', \phi; \mu, \eta_H) - \mathcal{V}_c^1(a', \phi; \mu, \eta_L)\big] - \big[\mathcal{V}_c^0(a', \phi; \mu, \eta_H) - \mathcal{V}_c^0(a', \phi; \mu, \eta_L)\big] \;\geq\; 0,$$

so the continuation-value component of $\mathcal{O}$ has (weakly) increasing differences in (I, η).

Combining (a) and (b), we conclude that, for every fixed (a, ϕ, x), the function

$$(I, \eta) \longmapsto \mathcal{O}(a, I; x, \phi; \eta)$$

has (weakly) increasing differences in (I, η) on $\{0,1\} \times \{\eta_L, \eta_H\}$.

Step 2: From the flow objective to the net value of innovation.

To avoid dependence of the feasible set on (I, η), extend $\mathcal{O}$ to all $x = (a', \mu) \in [a_{\min}, +\infty) \times \mathbb{R}_+$ by setting

$$\mathcal{O}(a, I; x, \phi; \eta) := -\infty \qquad \text{whenever} \qquad y(a, \phi; \eta) - \mathcal{E}(I, \mu, a'; \phi) < 0.$$

Under this convention Lemma 9 implies that, for each (a, ϕ, I, η),

$$V^I(a, \phi; \eta) = \sup_{x \in [a_{\min}, +\infty) \times \mathbb{R}_+} \mathcal{O}(a, I; x, \phi; \eta),$$

and that the supremum is in fact attained.

Since $\mathcal{O}(a, I; x, \phi; \eta)$ has increasing differences in (I, η) pointwise in x and the feasible set does not depend on (I, η), Milgrom and Shannon's (**?**) maximization theorem implies that $(I, \eta) \mapsto V^I(a, \phi; \eta)$ inherits (weakly) increasing differences. Hence, for each fixed (a, ϕ) the function

$$\eta \longmapsto \Delta(a, \phi; \eta) = V^1(a, \phi; \eta) - V^0(a, \phi; \eta)$$

is nondecreasing on $[\eta_L, \eta_H]$.

Step 3: Extensive-margin comparative statics.

Fix $\phi \geq \underline{\phi}$. From Step 2 we have, for every $a \geq a_{\min}$,

$$\Delta(a, \phi; \eta_H) \;\geq\; \Delta(a, \phi; \eta_L).$$

Therefore

$$\big\{a \geq a_{\min} : \Delta(a, \phi; \eta_L) \geq 0\big\} \;\subseteq\; \big\{a \geq a_{\min} : \Delta(a, \phi; \eta_H) \geq 0\big\}.$$

By Lemma 8, for each (ϕ, η) the function $a \mapsto \Delta(a, \phi; \eta)$ is increasing on the set where $I = 1$ is feasible. Together with the threshold characterization in Proposition 2, this inclusion implies

$$\underline{a}(\phi; \eta_H) \;\leq\; \underline{a}(\phi; \eta_L) \qquad \text{for all } \phi \geq \underline{\phi},$$

which proves part (i).

Step 4: Intensive-margin comparative statics.

Now fix a state (a, ϕ) such that

$$I^*(a, \phi; \eta_L) = I^*(a, \phi; \eta_H) = 1.$$

For each $\eta \in \{\eta_L, \eta_H\}$ define the conditional value of choosing R&D intensity $\mu \geq 0$ (given $I = 1$) by

$$H(a, \mu; \phi, \eta) := \sup_{a' \geq a_{\min}} \Big\{ U\big(y(a, \phi; \eta) - a' - \chi_v(\mu; \phi) - \chi_f(\phi)\big) + \beta\, \mathbb{E}\big[V(a', \phi'; \eta) \mid \phi, 1, \mu\big] \Big\},$$

with infeasible choices (those implying $c < 0$) assigned value $-\infty$. By continuity of the objective and compactness of the feasible set in a' induced by the budget constraint, the supremum is attained. For each (a, ϕ, η) the argmax correspondence

$$\mathcal{M}(a, \phi; \eta) := \arg\max_{\mu \geq 0} H(a, \mu; \phi, \eta)$$

is therefore nonempty and compact.

We claim that, for each fixed (a, ϕ), the function $H(a, \mu; \phi, \eta)$ has (weakly) increasing differences in (μ, η) on $[0, \infty) \times \{\eta_L, \eta_H\}$.

(a) Current-period utility. Fix (a', ϕ) and define

$$\mathcal{E}(\mu) := a' + \chi_v(\mu; \phi) + \chi_f(\phi), \qquad y(\eta) := y(a, \phi; \eta).$$

Since $\chi_v(\mu; \phi)$ is strictly increasing in μ, $\mathcal{E}(\mu)$ is strictly increasing. Lemma 14 (with $\tau \equiv \eta$) implies that $(\mathcal{E}, \eta) \mapsto U(y(\eta) - \mathcal{E})$ has increasing differences. By the standard composition result for increasing differences (see e.g., **?**), it follows that

$$(\mu, \eta) \longmapsto U\big(y(\eta) - \mathcal{E}(\mu)\big)$$

has increasing differences in (μ, η).

(b) Continuation value. For fixed (a', ϕ) define

$$\mathcal{V}_c^1(a', \mu; \phi, \eta) := \mathbb{E}\big[V(a', \phi'; \eta) \mid \phi, 1, \mu\big].$$

Let $\Delta_{a', \eta_H, \eta_L}$ be as in Step 1(b). Then

$$\mathcal{V}_c^1(a', \mu; \phi, \eta_H) - \mathcal{V}_c^1(a', \mu; \phi, \eta_L) = \mathbb{E}\big[\Delta_{a', \eta_H, \eta_L}(\phi') \mid \phi, 1, \mu\big].$$

By Lemma 18, $\Delta_{a',\eta_H,\eta_L}$ is nonnegative and nondecreasing on $[\underline{\phi}, +\infty)$ under $\epsilon = 0$. Lemma 5, property (Fμ+), states that conditional on $(\phi, I = 1)$, the distribution of ϕ' is strictly FOSD- and MLR-increasing in μ. Hence, for any $\mu_H > \mu_L \geq 0$,

$$\mathbb{E}\big[\Delta_{a',\eta_H,\eta_L}(\phi') \mid \phi, 1, \mu_H\big] \;\geq\; \mathbb{E}\big[\Delta_{a',\eta_H,\eta_L}(\phi') \mid \phi, 1, \mu_L\big],$$

with strict inequality whenever the reform changes the continuation value with positive probability along the induced future states. Equivalently,

$$\big[\mathcal{V}_c^1(a', \mu_H; \phi, \eta_H) - \mathcal{V}_c^1(a', \mu_L; \phi, \eta_H)\big] - \big[\mathcal{V}_c^1(a', \mu_H; \phi, \eta_L) - \mathcal{V}_c^1(a', \mu_L; \phi, \eta_L)\big] \;\geq\; 0,$$

so the continuation-value component exhibits (weakly) increasing differences in (μ, η).

Combining (a) and (b), we conclude that $H(a, \mu; \phi, \eta)$ has (weakly) increasing differences in (μ, η) for each fixed (a, ϕ).

By Milgrom and Shannon's (**?**) monotone selection theorem, the argmax correspondence $\eta \mapsto \mathcal{M}(a, \phi; \eta)$ is nondecreasing in η in the strong set order:

$$\min \mathcal{M}(a, \phi; \eta_H) \;\geq\; \min \mathcal{M}(a, \phi; \eta_L), \qquad \max \mathcal{M}(a, \phi; \eta_H) \;\geq\; \max \mathcal{M}(a, \phi; \eta_L).$$

On the interior, where $\mathcal{M}(a, \phi; \eta)$ is single-valued, this implies

$$\mu^*(a, \phi; \eta_H) \;\geq\; \mu^*(a, \phi; \eta_L)$$

whenever $I^*(a, \phi; \eta_L) = I^*(a, \phi; \eta_H) = 1$. This proves part (ii) and completes the proof of the proposition. □

OA-E Aggregation and TFP Decomposition

OA-E.1 Aggregation

The total capital and labor input used in the economy are given by:

$$K_t = N \cdot \Gamma(w) \int M(a, \phi; w)^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} dG_t(a, \phi), \tag{OA-49}$$

$$L_t = N \cdot \frac{\alpha_l}{\alpha_k w} \Gamma(w) \int M(a, \phi; w)^{-\frac{\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} dG_t(a, \phi), \tag{OA-50}$$

where $G_t(a, \phi)$ is the distribution of firms over states at time t. The aggregate output is given by:

$$
\begin{aligned}
Q_t =& \left[N \cdot \int \phi^{\tilde{\alpha}_\phi} k^*(a,\,\phi)^{\tilde{\alpha}_k} l^*(a,\,\phi)^{\tilde{\alpha}_l} dG_t(a, \phi)\right]^{\frac{\sigma}{\sigma-1}} \\
=& \left[N \cdot \Gamma(w)^{\tilde{\alpha}_k+\tilde{\alpha}_l} \left(\frac{\alpha_l}{\alpha_k w}\right)^{\tilde{\alpha}_l} \int \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} M(a,\,\phi)^{\frac{-\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} dG_t(a, \phi)\right]^{\frac{\sigma}{\sigma-1}} \\
\Rightarrow \quad Q_t =& \Gamma(w)^{\alpha_k+\alpha_l} \left(\frac{\alpha_l}{\alpha_k w}\right)^{\alpha_l} \left[\int \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} M(a,\,\phi)^{\frac{-\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} dG_t(a, \phi)\right]^{\frac{\sigma}{\sigma-1}} \qquad \text{(OA-51)}
\end{aligned}
$$

Then the actual TFP (ATFP) is given by:

$$
\begin{aligned}
\text{ATFP}_t = \frac{Q_t}{K_t^{\alpha_k} L_t^{\alpha_l}} =& N^{\frac{1}{\sigma-1}} \frac{\left[\int \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} M(a,\,\phi)^{\frac{-\tilde{\alpha}_k}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} dG_t(a, \phi)\right]^{\frac{\sigma}{\sigma-1}-\alpha_l}}{\left(\int M(a, \phi)^{-\frac{1-\tilde{\alpha}_l}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} \phi^{\frac{\tilde{\alpha}_\phi}{1-\tilde{\alpha}_k-\tilde{\alpha}_l}} dG_t(a, \phi)\right)^{\alpha_k}} \\
=& N^{\frac{1}{\sigma-1}} \frac{\left[\int \phi^{\sigma\tilde{\alpha}_\phi} M(a,\,\phi)^{-\sigma\tilde{\alpha}_k} dG_t(a, \phi)\right]^{\frac{\sigma}{\sigma-1}-\alpha_l}}{\left[\int \phi^{\sigma\tilde{\alpha}_\phi} M(a, \phi)^{-\sigma(1-\tilde{\alpha}_l)} dG_t(a, \phi)\right]^{\alpha_k}} \qquad \text{(OA-52)}
\end{aligned}
$$

In the absence of borrowing constraints, i.e., $M(a, \phi) = r_t + \delta$, the efficient TFP (ETFP) is given by:

$$
\text{ETFP}_t = N^{\frac{1}{\sigma-1}} \left(\int \phi^{(\sigma-1)\alpha_\phi} dG_t(a,\,\phi)\right)^{\frac{1}{\sigma-1}} \qquad \text{(OA-53)}
$$

The ETFP measures the aggregate productivity when the capital is allocated efficiently across firms given the fundamental productivity distribution. However, due to financial frictions, the actual allocation of capital deviates from the efficient one, leading to a gap between ATFP and ETFP.

OA-E.2 Aggregate TFP under Joint Log-normal Distribution of ϕ and $M(a,\,\phi)$

Suppose that ϕ and $M(a,\,\phi)$ are jointly log-normally distributed:

$$
\begin{pmatrix} \log \phi \\ \log M \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_\phi \\ \mu_M \end{pmatrix}, \begin{pmatrix} \sigma_\phi^2 & \sigma_{\phi M} \\ \sigma_{\phi M} & \sigma_M^2 \end{pmatrix}\right)
$$

Then we can derive a closed-form expression for the ATFP as follows:

$$
\text{ATFP}_t = N^{\frac{1}{\sigma-1}} \frac{\left[\int \phi^{\sigma\tilde{\alpha}_\phi} M(a,\,\phi)^{-\sigma\tilde{\alpha}_k} dG_t(a, \phi)\right]^{\frac{\sigma}{\sigma-1}-\alpha_l}}{\left[\int \phi^{\sigma\tilde{\alpha}_\phi} M(a, \phi)^{-\sigma(1-\tilde{\alpha}_l)} dG_t(a, \phi)\right]^{\alpha_k}}
$$

$$=N^{\frac{1}{\sigma-1}} \frac{\left[\exp(\sigma\tilde{\alpha}_\phi\mu_\phi - \sigma\tilde{\alpha}_k\mu_M + \frac{1}{2}(\sigma^2\tilde{\alpha}_\phi^2\sigma_\phi^2 + \sigma^2\tilde{\alpha}_k^2\sigma_M^2 - 2\sigma^2\tilde{\alpha}_\phi\tilde{\alpha}_k\sigma_{\phi M}))\right]^{\frac{\sigma}{\sigma-1}-\alpha_l}}{\left[\exp(\sigma\tilde{\alpha}_\phi\mu_\phi - \sigma(1-\tilde{\alpha}_l)\mu_M + \frac{1}{2}(\sigma^2\tilde{\alpha}_\phi^2\sigma_\phi^2 + \sigma^2(1-\tilde{\alpha}_l)^2\sigma_M^2 - 2\sigma^2\tilde{\alpha}_\phi(1-\tilde{\alpha}_l)\sigma_{\phi M}))\right]^{\alpha_k}}$$

and

$$\begin{aligned}
\textit{coefficient of } \mu_\phi =& \sigma\tilde{\alpha}_\phi \left(\frac{\sigma}{\sigma-1} - \alpha_l - \alpha_k\right) = \alpha_\phi, \\
\textit{coefficient of } \mu_M =& -\sigma\left(\tilde{\alpha}_k\left(\frac{\sigma}{\sigma-1} - \alpha_l\right) - (1-\tilde{\alpha}_l)\alpha_k\right) = 0, \\
\textit{coefficient of } \sigma_\phi^2 =& \frac{1}{2}\sigma^2\tilde{\alpha}_\phi^2\left(\frac{\sigma}{\sigma-1} - \alpha_l - \alpha_k\right) = \frac{\sigma^2}{2(\sigma-1)}\tilde{\alpha}_\phi^2, \\
\textit{coefficient of } \sigma_M^2 =& \frac{1}{2}\sigma^2\left(\tilde{\alpha}_k^2\left(\frac{\sigma}{\sigma-1} - \alpha_l\right) - (1-\tilde{\alpha}_l)^2\alpha_k\right) \\
=& \frac{\sigma^2}{2}\alpha_k(1-\tilde{\alpha}_l)\left[\tilde{\alpha}_k - (1-\tilde{\alpha}_l)\right] = -\frac{\alpha_k(1-\sigma\tilde{\alpha}_k)}{2}, \\
\textit{coefficient of } \sigma_{\phi M} =& -\sigma^2\tilde{\alpha}_\phi\left(\tilde{\alpha}_k\left(\frac{\sigma}{\sigma-1} - \alpha_l\right) - (1-\tilde{\alpha}_l)\alpha_k\right) = 0.
\end{aligned}$$

Therefore, the ATFP can be expressed as:

$$\text{ATFP}_t = N^{\frac{1}{\sigma-1}} \exp\left(\alpha_\phi\mu_\phi + \frac{\sigma^2}{2(\sigma-1)}\tilde{\alpha}_\phi^2\sigma_\phi^2 - \frac{\alpha_k(1-\sigma\tilde{\alpha}_k)}{2}\sigma_M^2\right) \tag{OA-54}$$

Note that when ϕ_i is a log-normal distribution, the efficient TFP is given by:

$$\text{ETFP}_t = N^{\frac{1}{\sigma-1}} \exp\left(\alpha_\phi\mu_\phi + \frac{\sigma^2}{2(\sigma-1)}\tilde{\alpha}_\phi^2\sigma_\phi^2\right). \tag{OA-55}$$

Therefore, ATFP can be re-expressed as:

$$\text{ATFP}_t = \text{ETFP}_t \cdot \exp\left(-\frac{\alpha_k(1-\sigma\tilde{\alpha}_k)}{2}\sigma_M^2\right) < \text{ETFP}_t \tag{OA-56}$$

OA-F Quantitative Analysis Appendix

OA-F.1 Algorithms

OA-F.1.1 Solving for the Firm's Optimal Problem at Steady State

We solve for the firm's optimal problem at the steady state (where the aggregate wage is constant, $w_t = w_{ss}$) using Value Function Iteration (VFI). The solution algorithm proceeds in two main phases: solving the static intratemporal problem and solving the dynamic intertemporal problem.

1. Discretization and Static Optimization First, we discretize the state space for productivity and assets. Let the grid for idiosyncratic productivity be $\Phi = \{\phi_1, \phi_2, ..., \phi_{n_\phi}\}$ and the grid for assets be $\mathcal{A} = \{a_1, a_2, ..., a_{n_a}\}$.

Before entering the VFI loop, we solve for the static optimal labor and capital decisions to obtain the operating profit function $\pi(a, \phi; w)$. Given the wage w, the optimal labor input l^* is determined by the first-order condition:

$$l^*(k, \phi; w) = \left[\frac{\alpha_l \frac{\sigma-1}{\sigma}}{w} \left(\phi^{\alpha_\phi} k^{\alpha_k}\right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{1-\alpha_l \frac{\sigma-1}{\sigma}}}$$

Substituting l^* into the revenue function, we define the auxiliary term $\Lambda(\phi; w)$ representing the firm's revenue capacity per unit of effective capital:

$$\Lambda(\phi; w) = \left(1 - \alpha_l \frac{\sigma - 1}{\sigma}\right) \left(\frac{\alpha_l \frac{\sigma-1}{\sigma}}{w}\right)^{\frac{\alpha_l \frac{\sigma-1}{\sigma}}{1-\alpha_l \frac{\sigma-1}{\sigma}}} \phi^{\frac{\alpha_\phi \frac{\sigma-1}{\sigma}}{1-\alpha_l \frac{\sigma-1}{\sigma}}}$$

Let $\hat{\alpha}_k = \frac{\alpha_k(\sigma-1)/\sigma}{1-\alpha_l(\sigma-1)/\sigma}$. The firm chooses capital k to maximize static profits subject to the financial constraint:

$$\max_k \quad \Lambda(\phi; w) k^{\hat{\alpha}_k} - (r + \delta)k \quad \text{s.t.} \quad k \leq \theta a + \eta\phi$$

The optimal capital choice $k^*(a, \phi)$ is:

$$k^*(a, \phi) = \min \left\{ \left[\frac{\hat{\alpha}_k \Lambda(\phi; w)}{r + \delta} \right]^{\frac{1}{1-\hat{\alpha}_k}}, \ \theta a + \eta\phi \right\}$$

Consequently, the maximized operating profit $\pi(a, \phi)$ is computed and stored for all grid points (a_i, ϕ_j).

2. Dynamic Programming We solve the Bellman equation iteratively:

1. **Initialization:** We guess an initial value function $V^{(0)}(a_i, \phi_j)$ for all grid points. We use two-dimensional linear interpolation to evaluate the value function $V^{(0)}(a', \phi')$ off the grid points.

2. **Optimization Step:** In iteration k, for each state (a_i, ϕ_j), we solve for the optimal next-period assets a' and R&D intensity μ. The optimization is performed via a nested Golden Section Search:

- *Inner Loop:* Given a candidate a', we solve for the optimal $\mu^*(a')$.
- *Outer Loop:* We maximize over a' taking the dependence of μ on a' into account.

The conditional expectation $\mathbb{E}[V^{(k)}(a', \phi')]$ is computed using numerical integration (Gauss-Hermite quadrature) over the distribution of the innovation shock ξ.

3. **Howard's Improvement (Policy Iteration):** To accelerate convergence, after obtaining the optimal policies $a'^{(k)}(a, \phi)$ and $\mu^{(k)}(a, \phi)$, we update the value function by iterating on the Bellman equation multiple times while holding these policies fixed.

4. **Convergence:** We compute the new value function $V^{(k+1)}$. We check the sup-norm distance:
$$\epsilon = \max_{i,j} \left| V^{(k+1)}(a_i, \phi_j) - V^{(k)}(a_i, \phi_j) \right|$$
If $\epsilon <$ Tolerance (e.g., 10^{-6}), the algorithm terminates. Otherwise, we set $V^{(k)} = V^{(k+1)}$ and repeat from Step 2.

OA-F.1.2 Solving for the Equilibrium Transition Path

To solve for the transition dynamics from an initial steady state (pre-policy) to a final steady state (post-policy), we employ a shooting method to determine the equilibrium wage path $\{w_t\}_{t=1}^T$. The algorithm proceeds as follows:

1. **Solve Steady States:**
   - We solve for the initial steady state distribution $\lambda_0(a, \phi)$ and wage $w_{initial}$.
   - We solve for the final steady state wage w_{final} and value function V_{final} using the method described in Section OA-F.1.1. This involves finding the w that clears the labor market using a non-linear solver (`fzero`).

2. **Initial Guess:** We guess a transition path for wages, $\mathbf{w}^{(0)} = \{w_1^{(0)}, w_2^{(0)}, ..., w_T^{(0)}\}$. Typically, this is initialized as a linear interpolation between $w_{initial}$ and w_{final}.

3. **Backward Induction:** Given the wage path $\mathbf{w}^{(i)}$, we solve for the value functions and policy functions backward in time from $t = T - 1$ to $t = 1$.
   - At $t = T$, $V_T(\cdot) = V_{final}(\cdot)$.

- For $t = T-1, \ldots, 1$, we solve the Bellman equation given $w_t^{(i)}$ and the continuation value $V_{t+1}(\cdot)$. This yields the time-dependent policy functions for assets $a'_t(a, \phi)$ and R&D $\mu_t(a, \phi)$.

4. **Forward Simulation and Market Clearing:** We simulate the economy forward from $t = 1$ to T to update the wage path.

   - We start with the initial distribution λ_0.
   - For each period $t = 1, \ldots, T$:
     (a) We calculate the aggregate labor supply based on the current distribution λ_{t-1} and the policy functions derived in the backward step.
     (b) We find the equilibrium wage $\tilde{w}_t$ that clears the labor market at period t. This is done by solving for the wage such that total labor demand equals the fixed labor supply, using the non-stochastic simulation method proposed by **?** to track the distribution λ_t precisely without Monte Carlo sampling noise.
     (c) We update the distribution to λ_t using the transition matrix implied by the policy functions.

5. **Update and Convergence:** We compare the implied market-clearing wages $\tilde{\mathbf{w}}$ derived in the forward step with the guessed path $\mathbf{w}^{(i)}$. If $||\tilde{\mathbf{w}} - \mathbf{w}^{(i)}|| < 10^{-5}$, the equilibrium path is found. Otherwise, we update the guess using a convex combination:

$$\mathbf{w}^{(i+1)} = \gamma \tilde{\mathbf{w}} + (1-\gamma)\mathbf{w}^{(i)}$$

   where $\gamma \in (0, 1)$ is a dampening parameter, and return to Step 3.

OA-F.1.3 Simulated Method of Moments (SMM) Estimation

We employ a two-step Simulated Method of Moments (SMM) approach to estimate the model parameters.

Step 1: Estimation of Structural Parameters First, we estimate the structural parameters governing firm dynamics and R&D decisions using data from the pre-policy period (2005–2007). During this step, we set the patent collateral parameter $\eta = 0$, reflecting the absence of the policy. The estimation procedure proceeds as follows:

1. **Simulation:** We simulate a panel of $N = 10,000$ firms over $T = 1,100$ periods. Initial firm states $\{a_{i,1}, \phi_{i,1}\}$ are drawn randomly.

2. **Solving the Dynamic Problem:** For each candidate parameter vector, we solve the firm's infinite-horizon dynamic optimization problem. We employ Value Function Iteration (VFI) on a discretized state space for productivity (ϕ) and net worth (a). To ensure precision and mitigate grid-dependence, we allow for continuous policy choices (net worth a' and R&D target μ). Specifically, we approximate the expected value function using linear interpolation and employ a Golden Section Search algorithm to determine the optimal policy functions. See details in Section OA-F.1.1.

3. **Moment Calculation:** To eliminate the influence of initial conditions and ensure the economy reaches its ergodic distribution, we discard the first 1,000 periods as a burn-in. We use the remaining 100 periods to compute the simulated moments, applying the same definitions and sampling restrictions used to construct the empirical data moments.

Step 2: Estimation of Policy Parameter (η_1) In the second step, we fix the structural parameters estimated above and estimate the policy parameter η_1 by targeting the treatment effects observed in our Difference-in-Differences (DID) regression.

- We simulate the transition dynamics starting from the steady state obtained in Step 1.
- For a given candidate value of η_1, firms re-optimize their decision rules accounting for the new collateral constraint relaxation. We solve for the new policy functions while holding the general equilibrium prices fixed.
- We simulate the economy forward for 6 periods. We calculate the average change in the fraction of firms undertaking R&D from periods $t = 4$ to $t = 6$ relative to the pre-policy steady state. This constructed moment is matched against the coefficient from our DID regression.

This two-step approach allows us to isolate the identification of η_1.

Objective Function and Asymptotics Let Θ denote the vector of parameters to be estimated. The SMM estimator $\hat{\Theta}$ minimizes the weighted distance between the empirical moments

(M^d) and the simulated moments ($M^s(\Theta)$):

$$\hat{\Theta} = \arg\min_{\Theta} \left[M^d - M^s(\Theta)\right]' W \left[M^d - M^s(\Theta)\right], \tag{OA-57}$$

where W is a positive-definite weighting matrix. We employ the inverse of the variance-covariance matrix of the data moments, calculated via block bootstrap with 1,000 replications, as the weighting matrix ($W = \hat{\Omega}^{-1}$). To locate the global minimum, we combine a pattern search algorithm (to explore the parameter space) with the Nelder-Mead simplex algorithm (‘fminsearch‘) for final convergence.

Under standard regularity conditions, the estimator $\hat{\Theta}$ is asymptotically normal:

$$\sqrt{N}\left(\hat{\Theta} - \Theta_0\right) \xrightarrow{d} \mathcal{N}(0, \mathbf{V}),$$

where the asymptotic variance-covariance matrix $\mathbf{V}$ is given by:

$$\mathbf{V} = \left(1 + \frac{1}{S}\right)(J'WJ)^{-1}J'W\Omega WJ(J'WJ)^{-1}.$$

Here, S denotes the ratio of the number of simulated observations to data observations, and Ω is the asymptotic covariance matrix of the empirical moments. J represents the Jacobian matrix of the simulated moments with respect to the parameters, $J = \nabla_{\Theta} M^s(\Theta)$.

We estimate the Jacobian $\hat{J}$ using a two-sided finite difference approximation. The element corresponding to the i-th moment and j-th parameter is calculated as:

$$\hat{J}_{i,j} \approx \frac{M_i^s(\hat{\Theta} + h^j e_j) - M_i^s(\hat{\Theta} - h^j e_j)}{2h^j},$$

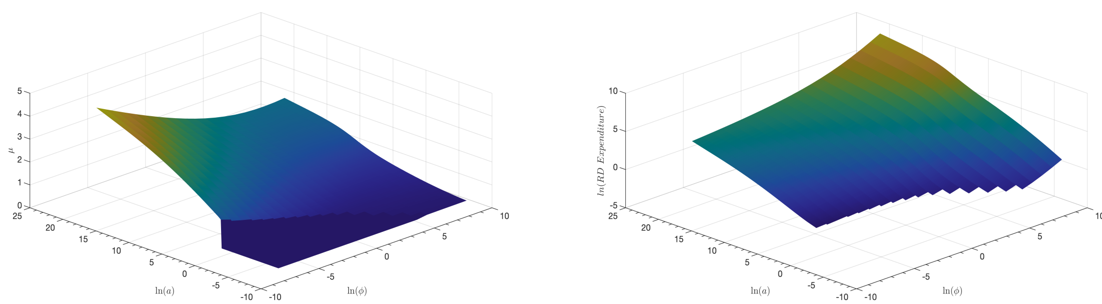
where e_j is the standard basis vector and h^j is the step size.

OA-F.2 R&D and Savings Policy Functions

Policy Function of R&D Investment Given the estimated model, we examine the properties of the policy functions for the R&D target (μ) and R&D expenditure (RDX). Figure (OA-2) displays these policy functions. The R&D policy function reveals that firms with substantially low net worth and productivity generally undertake no R&D investments, primarily due to binding financial constraints. Among firms that do invest, holding productivity constant, those with greater net worth target higher productivity growth rates (i.e., larger μ).

However, the relationship between productivity and the R&D target is more nuanced and is not clearly monotonic. This is because two opposing forces related to productivity drive R&D decisions. On one hand, given $\zeta_2 > 0$, higher productivity firms face higher R&D costs. This creates a convergence or "catching up" effect between low- and high-productivity firms, particularly when both possess high net worth. On the other hand, higher productivity firms generate larger cash flows; this results in higher consumption and a lower marginal utility cost of investment, which incentivizes further R&D.

The corresponding R&D expenditure policy function, depicted on the right, further elucidates these investment patterns. Conditional on net worth, firms with higher productivity incur greater R&D costs despite targeting lower expected growth rates (i.e., smaller μ). This phenomenon underscores that frontier firms typically incur disproportionately higher costs to sustain equivalent rates of productivity growth, an effect driven by the constraint $\zeta_2 > 0$.

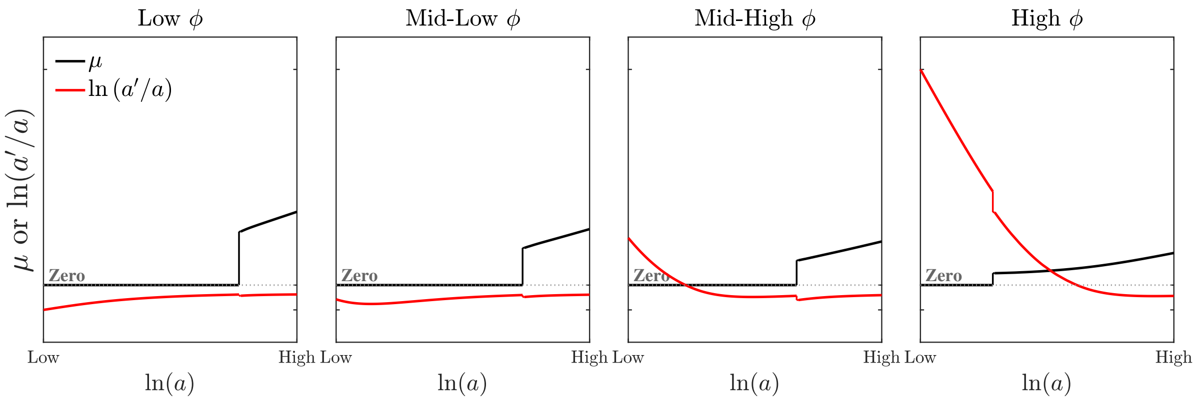


(a) μ (b) RDX

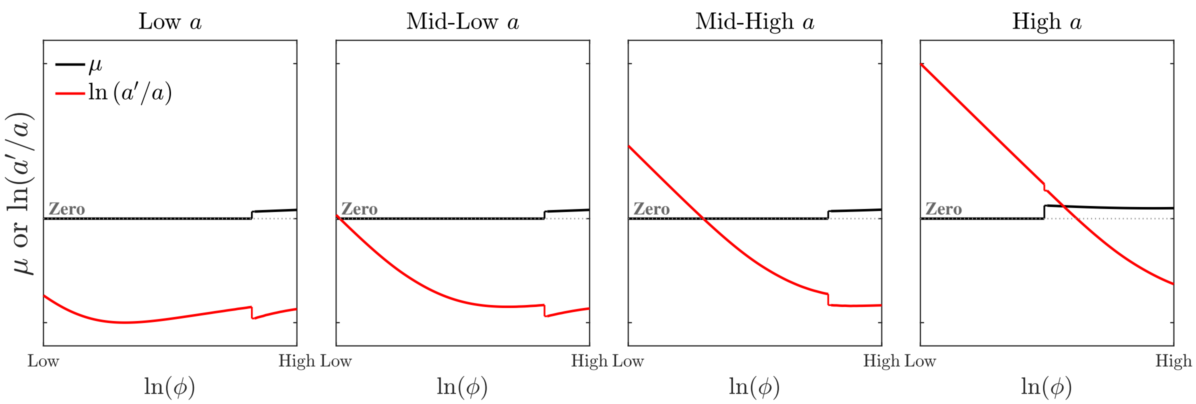
Figure OA-2: Policy Function of R&D Target and R&D Expenditure

Policy Functions Related to Propositions 2 and 3 The figures below illustrate the policy functions described in Propositions 2 and 3.

Figure OA-3a displays the optimal R&D and saving decisions as functions of net worth (a) across different productivity levels, while Figure OA-3b displays these decisions as functions of productivity (ϕ) across different net worth levels.



(a) Optimal R&D and Saving as Functions of Net Worth across Different Productivity Levels

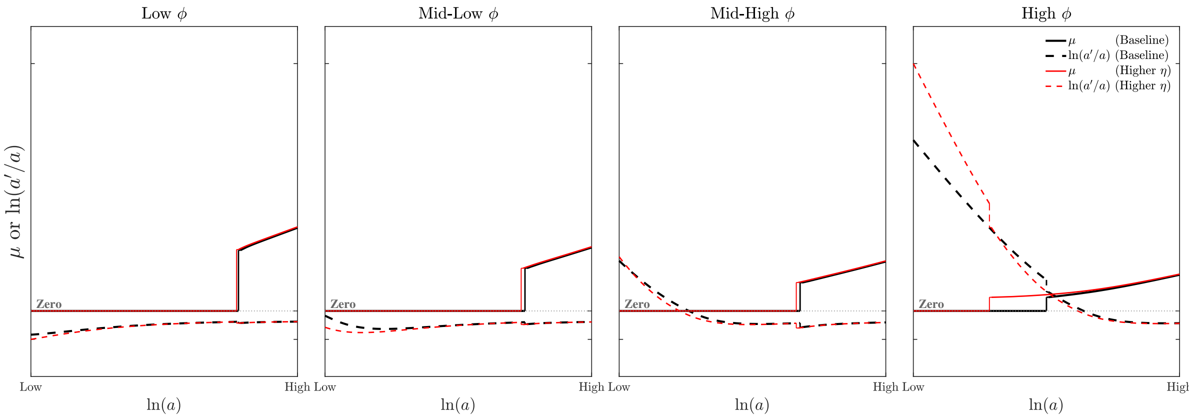


(b) Optimal R&D and Saving as Functions of Productivity across Different Net Worth Levels

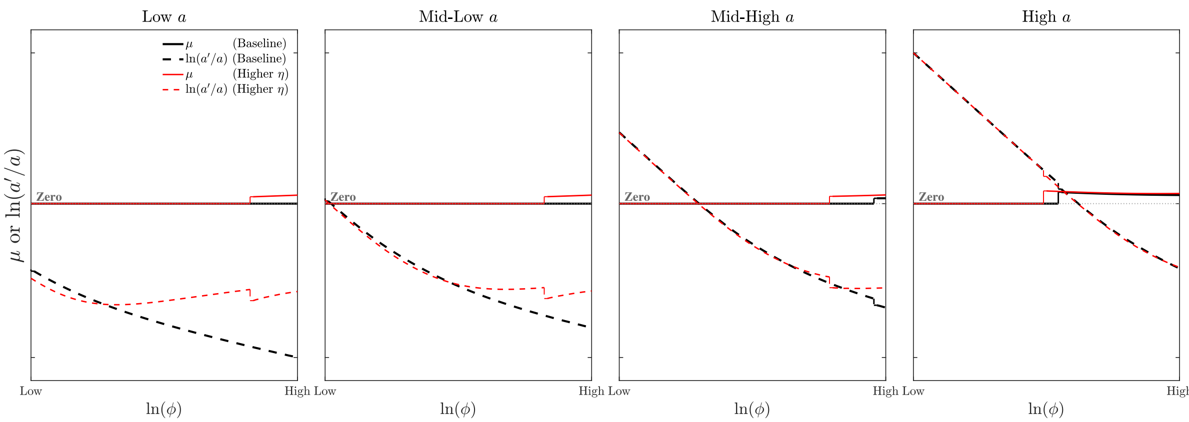
Figure OA-3: Optimal R&D and Saving: Baseline

Note: the scales of y-axis for μ and $\ln(a'/a)$ are the same.

Building on Figure OA-3a, Figure OA-4a illustrates the optimal R&D and saving decisions as functions of net worth (a) when η is higher. Similarly, Figure OA-4b depicts these decisions as functions of productivity (ϕ) under the higher η value, corresponding to the baseline in Figure OA-3b.



(a) Optimal R&D and Saving as Functions of Net Worth across Different Productivity Levels

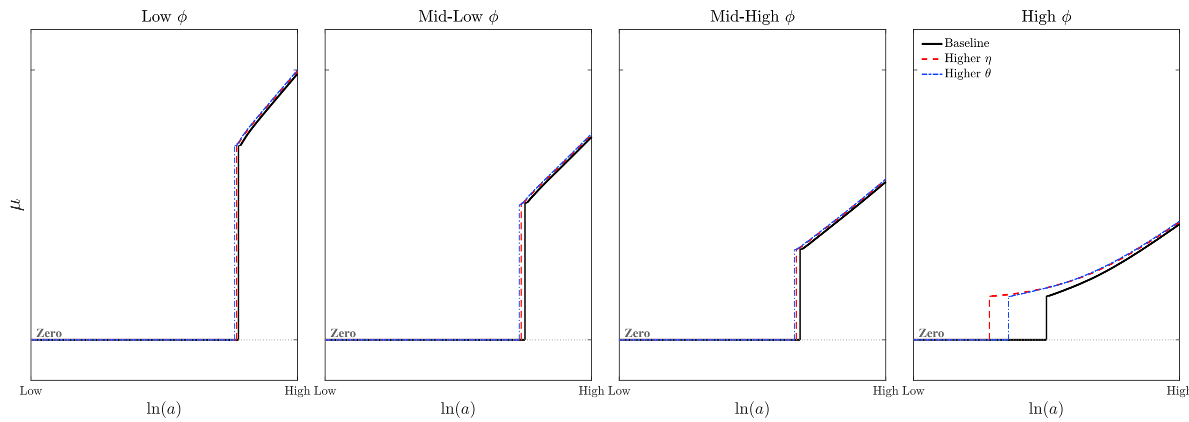


(b) Optimal R&D and Saving as Functions of Productivity across Different Net Worth Levels

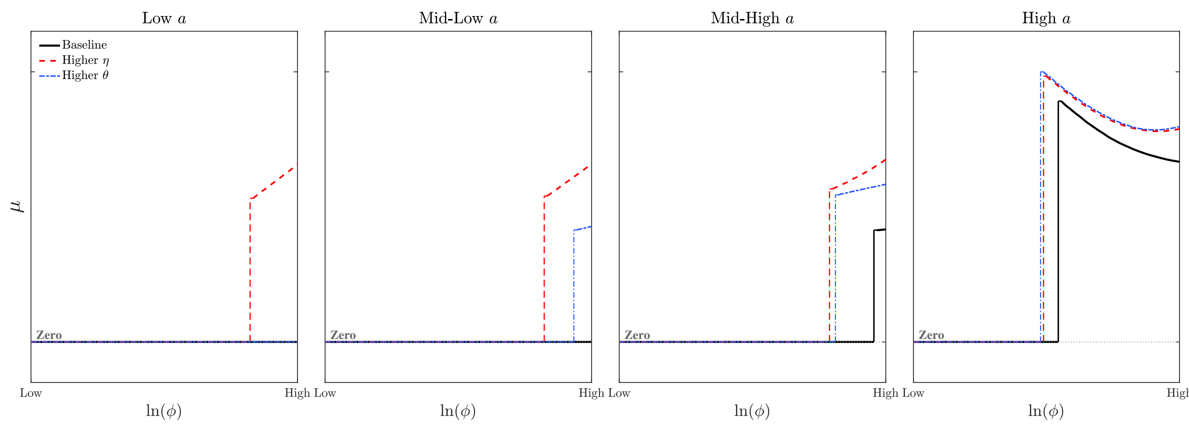
Figure OA-4: Optimal R&D and Saving: Baseline VS IP-backed Lending Policy

Note: the scales of y-axis for μ and $\ln(a'/a)$ are the same.

Figure OA-5 illustrates the impact of changes in θ and η on R&D target policy functions. The results indicate that increasing η has a greater impact on firms with high ϕ and relatively low a. Conversely, increasing θ exerts a more significant effect on firms with high a and relatively low ϕ.



(a) Optimal R&D Targets as Functions of Net Worth across Different Productivity Levels



(b) Optimal R&D Targets as Functions of Productivity across Different Net Worth Levels

Figure OA-5: The Impact of θ and η on R&D Targets

OA-G Extension: R&D Spillovers

In this section, we extend the baseline model presented in the main text to incorporate long-run economic growth driven by R&D spillovers. To ensure a balanced growth path, we first specify the functional form of the collateral value of intellectual properties. Building on this adjusted model, we then introduce the spillover effects of R&D. Finally, we re-estimate the model and quantify the impact of the policy on static gains and long-run growth.

OA-G.1 A Stationary Model with a Specified Function of Collateral Value of IP

We now specify the baseline model with a specific functional form for the collateral value of IP. This specification ensures that the baseline model—which abstracts from aggregate growth as in the main text—is isomorphic to the detrended model with long-run economic growth presented later. The firm's optimization problem at the steady state is defined recursively as follows:

$$V(a,\phi)=\max_{\{c,a',\mu\}}\left\{\ln(c)+\beta\mathbb{E}[V(a',\phi')|\phi,\mu]\right\}$$

subject to the budget constraint:

$$c+\chi(\mu;\,\phi)+a'=\pi(a,\phi;w)+(1+r)a,$$

where the profit function is given by:

$$\pi(a,\phi;w)=\max_{k,l}\left\{(\phi^{\alpha_\phi}k^{\alpha_k}l^{\alpha_l})^{\frac{\sigma-1}{\sigma}}-wl-(r+\delta)k\right\},$$

subject to the collateral constraint:

$$k\leq\theta a+\eta\Psi(\phi).$$

The productivity process evolves according to:

$$\ln(\phi')=\rho\ln(\phi)+\mu+\bar{\mu}+\sigma_\xi\xi',$$

where the collateral value of intangible assets is:

$$\Psi(\phi)=\phi^{\alpha_\phi/(\frac{\sigma}{\sigma-1}-\alpha_k)}.$$

The equilibrium wage rate w clears the labor market such that total labor demand equals total supply $\bar{L}$.

OA-G.2 A Model with R&D Spillovers

In this section, we specify the extended model incorporating spillovers. We assume that the average productivity in the economy generates a spillover effect on firm-level productivity. We then derive the detrended system and demonstrate that, under the specified functional forms, the detrended firm problem is isomorphic to the baseline stationary problem. This isomorphism allows us to employ the baseline estimation strategy. We focus our quantitative analysis on the balanced growth path (BGP) of this extended economy.

OA-G.2.1 The Extended Environment

Let variables with a tilde $(\tilde{\cdot})$ denote values in the growing economy. The firm's problem is:

$$\max_{\{\tilde{c}_{i,t},\tilde{a}_{i,t+1},\tilde{\mu}_{i,t}\}} \sum_{t=0}^{\infty} \beta^t \ln(\tilde{c}_{i,t})$$

subject to:

$$\tilde{c}_{i,t} + \tilde{\chi}(\tilde{\mu}_{i,t}; \tilde{\phi}_{i,t}) + \tilde{a}_{i,t+1} = \tilde{\pi}(\tilde{a}_{i,t}, \tilde{\phi}_{i,t}; \tilde{w}_t) + (1 + \tilde{r})\tilde{a}_{i,t},$$

where the period-profit maximization problem is:

$$\tilde{\pi}(\tilde{a}_{i,t}, \tilde{\phi}_{i,t}; \tilde{w}_t) = \max_{\{\tilde{k}_{i,t},\tilde{l}_{i,t}\}} \left[\left(\tilde{\phi}_{i,t}^{\alpha_\phi} \tilde{k}_{i,t}^{\alpha_k} \tilde{l}_{i,t}^{\alpha_l} \right)^{\frac{\sigma-1}{\sigma}} - \tilde{w}_t \tilde{l}_{i,t} - (\tilde{r} + \tilde{\delta})\tilde{k}_{i,t} \right],$$

subject to the extended collateral constraint:

$$\tilde{k}_{i,t} \leq \tilde{\theta}\tilde{a}_{i,t} + \tilde{\eta}\tilde{\Psi}(\tilde{\phi}_{i,t}).$$

The law of motion for productivity includes an aggregate spillover term, $\overline{\ln \phi}_t$:

$$\ln(\widetilde{\phi}_{i,t+1}) = \rho \ln(\widetilde{\phi}_{i,t}) + \tilde{\mu}_{i,t} + (1 - \rho)\overline{\ln \phi}_t + \bar{\mu} + \sigma_\xi \xi_{i,t+1}, \tag{OA-58}$$

where $\overline{\ln \phi}_t = \int_i \ln(\widetilde{\phi}_{i,t})di$. This specification implies that the growth rate of average productivity is driven by R&D choices:

$$\overline{\ln \phi}_t - \overline{\ln \phi}_{t-1} = \int_i \tilde{\mu}_{i,t-1}di + \bar{\mu} \equiv g_{\phi,t-1}.$$

Balanced Growth Path. A balanced growth path (BGP) is a competitive equilibrium path along which aggregate quantities, wages, and the trend variable X_t grow at a constant rate g, while the interest rate $\tilde{r}$ and labor supply remain constant. Formally, a BGP satisfies the following conditions:

1. **Constant Aggregate Growth:** The growth rate of the economy, $1 + g \equiv X_t/X_{t-1}$, is constant and determined by the stationary average R&D intensity $\mathbb{E}[\tilde{\mu}_{i,t}] = \mu^*$ such that:

$$1 + g = \left[\exp\left(\mu^* + \bar{\mu}\right)\right]^{\vartheta}, \quad \text{where } \vartheta \equiv \frac{\alpha_\phi}{\frac{\sigma}{\sigma-1} - \alpha_k}. \tag{OA-59}$$

2. **Stationarity of Detrended Variables:** The joint distribution of detrended firm-level state variables, assets $\hat{a}_{i,t} \equiv \tilde{a}_{i,t}(1 + g)/X_t$ and relative productivity $\widehat{\ln \phi}_{i,t} = \ln \widetilde{\phi}_{i,t} - \overline{\ln \phi}_t - \frac{1}{1-\rho}(-\mathbb{E}\tilde{\mu} - \bar{\mu})$, is time-invariant.

3. **Policy Consistency:** Given the stationary distribution and prices, the optimal firm policies for labor $\hat{l}_{i,t}$, R&D investment $\hat{\mu}_{i,t}$, and detrended capital $\hat{k}_{i,t}$ are time-invariant functions of the state $(\hat{a}_{i,t}, \widehat{\ln\phi}_{i,t})$.

4. **Market Clearing:** The labor market clears at the fixed supply $\bar{L}$, and the aggregate resource constraint holds for the growing economy.

OA-G.2.2 Balanced Growth and Detrending

We show that the detrended economy can map to our baseline model in Section OA-G.1. Let's define the scaling factor ϑ as:

$$\vartheta \equiv \frac{\alpha_\phi}{\frac{\sigma}{\sigma-1} - \alpha_k}.$$

We posit that aggregate output, capital, and wages grow at a rate g determined by the evolution of aggregate productivity. We define the trend variable X_t:

$$X_t \equiv \left[\exp(\overline{\ln\phi}_t)\right]^{\vartheta}.$$

The growth rate of the economy is $1+g = X_t/X_{t-1}$. At the BGP, $\mathbb{E}\tilde{\mu}$ is constant, implying a constant growth rate:

$$1+g = \left[\exp(\mathbb{E}\tilde{\mu} + \bar{\mu})\right]^{\vartheta}.$$

We define the following detrended variables (denoted by hats):

- Productivity: $\widehat{\ln\phi}_{i,t} = \ln\widetilde{\phi}_{i,t} - \overline{\ln\phi}_t - \frac{1}{1-\rho}(-\mathbb{E}\tilde{\mu} - \bar{\mu})$.
- Assets: $\hat{a}_{i,t} = \frac{\tilde{a}_{i,t}}{X_t}(1+g)$.
- Capital and Consumption: $\hat{k}_{i,t} = \frac{\tilde{k}_{i,t}}{X_t}, \quad \hat{c}_{i,t} = \frac{\tilde{c}_{i,t}}{X_t}$.
- Labor and R&D Choice: $\hat{l}_{i,t} = \tilde{l}_{i,t}, \quad \hat{\mu}_{i,t} = \tilde{\mu}_{i,t}$.
- Wage: $\hat{w} = \frac{\tilde{w}_t}{X_t}\left\{\exp\left[\frac{1}{1-\rho}(-\mathbb{E}\tilde{\mu} - \bar{\mu})\right]\right\}^{-\frac{\alpha_\phi}{\alpha_l}}$.

Proposition 8. Assume the following parameter mappings between the extended and baseline models:

1. Interest rates: $1+\tilde{r} = (1+r)(1+g)$ and $\tilde{r} + \tilde{\delta} = r + \delta$.

2. Collateral parameters: $\tilde{\theta} = (1+g)\theta$ and $\tilde{\eta} = \eta$.

3. Collateral function: $\tilde{\Psi}(x) = \left(\frac{x}{\exp\left[\frac{1}{1-\rho}(-\mathbb{E}\tilde{\mu}-\bar{\mu})\right]}\right)^{\vartheta}$.

4. R&D Cost function $\tilde{\chi}(\cdot;\cdot)$ scales with X_t such that $\tilde{\chi}(\tilde{\mu};\tilde{\phi}) = \chi(\hat{\mu};\hat{\phi})X_t$.

Then, the solution to the firm's problem in the extended model along the BGP is isomorphic to the solution of the baseline stationary model in Subsection OA-G.1.

Proof. We verify that the detrended variables satisfy the recursive problem of the baseline model.

1. The Productivity Process. Substituting the definition of $\widehat{\ln\phi}_{i,t}$ into (OA-58):

$$\widehat{\ln\phi}_{i,t+1} + \overline{\ln\phi}_{t+1} + C = \rho(\widehat{\ln\phi}_{i,t} + \overline{\ln\phi}_t + C) + \hat{\mu}_{i,t} + (1-\rho)\overline{\ln\phi}_t + \bar{\mu} + \sigma_\xi\xi_{i,t+1},$$

where $C = \frac{1}{1-\rho}(-\mathbb{E}\tilde{\mu} - \bar{\mu})$. Using the identity $\overline{\ln\phi}_{t+1} - \overline{\ln\phi}_t = \mathbb{E}\tilde{\mu} + \bar{\mu}$, the trend terms cancel, yielding the stationary process:

$$\widehat{\ln\phi}_{i,t+1} = \rho\widehat{\ln\phi}_{i,t} + \hat{\mu}_{i,t} + \bar{\mu} + \sigma_\xi\xi_{i,t+1}.$$

2. The Collateral Constraint. Substituting the definitions into $\tilde{k}_{i,t} \leq \tilde{\theta}\tilde{a}_{i,t} + \tilde{\eta}\tilde{\Psi}(\tilde{\phi}_{i,t})$:

$$\hat{k}_{i,t}X_t \leq (1+g)\theta\left[\frac{\hat{a}_{i,t}X_t}{1+g}\right] + \eta\left(\frac{\hat{\phi}_{i,t}\exp(\overline{\ln\phi}_t)\exp(C)}{\exp(C)}\right)^{\vartheta}.$$

Using $X_t = [\exp(\overline{\ln\phi}_t)]^{\vartheta}$ and dividing by X_t yields:

$$\hat{k}_{i,t} \leq \theta\hat{a}_{i,t} + \eta\left(\hat{\phi}_{i,t}\right)^{\vartheta}.$$

This matches the baseline constraint, with $\Psi(\hat{\phi}) = \hat{\phi}^{\vartheta}$.

3. The Budget Constraint. Consider the extended budget constraint. Substituting the detrended variables:

$$\hat{c}_{i,t}X_t + \chi(\hat{\mu}_{i,t};\hat{\phi}_{i,t})X_t + \hat{a}_{i,t+1}\frac{X_{t+1}}{1+g} = \tilde{\pi}(\cdot) + (1+\tilde{r})\hat{a}_{i,t}\frac{X_t}{1+g}.$$

Using $X_{t+1}/X_t = 1+g$ and $(1+\tilde{r})/(1+g) = 1+r$, and dividing by X_t:

$$\hat{c}_{i,t} + \chi(\hat{\mu}_{i,t}; \hat{\phi}_{i,t}) + \hat{a}_{i,t+1} = \frac{\tilde{\pi}(\cdot)}{X_t} + (1+r)\hat{a}_{i,t}.$$

The R&D scaling $\tilde{\chi}(\tilde{\mu}; \tilde{\phi}) = \chi(\hat{\mu}; \hat{\phi})X_t$ holds by substituting $\tilde{\phi}_{i,t} = \hat{\phi}_{i,t}\exp(\overline{\ln\phi}_t + C)$ into the original RDX form and noting the exponent ϑ matches X_t.[57] The profit function is homogeneous of degree 1 in X_t given $\hat{w}$'s definition and production exponents, so $\tilde{\pi}(\cdot)/X_t = \pi(\hat{a}, \hat{\phi}; \hat{w})$. Thus, the budget constraint matches.

4. The Objective Function. The firm maximizes $\sum \beta^t \ln(\tilde{c}_{i,t})$. Substituting $\tilde{c}_{i,t} = \hat{c}_{i,t}X_t$:

$$\sum_{t=0}^{\infty} \beta^t \ln(\hat{c}_{i,t}X_t) = \sum_{t=0}^{\infty} \beta^t \ln(\hat{c}_{i,t}) + \sum_{t=0}^{\infty} \beta^t \ln(X_t).$$

Since X_t grows at constant rate g, the second term is a constant $\Xi = \frac{\ln(X_0)}{1-\beta} + \frac{\beta\ln(1+g)}{(1-\beta)^2}$, which does not affect optimization. $\square$

Finally, the labor market clearing condition $\int \tilde{l}_{i,t}di = \bar{L}$ holds, implying $\int \hat{l}_{i,t}di = \bar{L}$, as labor is not detrended. Consequently, solving the detrended model is equivalent to solving the baseline model in Section OA-G.1.

OA-G.3 Calibration and Counterfactual Analysis

Having established the equivalence between the detrended BGP and the baseline model (with the modify function of $\Psi(\cdot)$), we proceed to estimate the extended model. The primary difference lies in the collateral function $\Psi(\cdot)$, which is a power function in the extended model (to satisfy BGP conditions) rather than linear. This requires re-estimating the pledgeability parameter η.

[57] Specifically, if the detrended R&D cost function is specified as in (33), then the original R&D cost function is

$$\begin{aligned}
\tilde{\chi}(\tilde{\mu}; \tilde{\phi}) &= \chi(\hat{\mu}; \hat{\phi})X_t \\
&= \begin{pmatrix} \left[exp\left(\widehat{ln\phi}_{i,t}\right)\right]^{\zeta_1} \times f \times \mathbb{1}\{\mu_{i,t} > 0\} + \\ \left[exp\left(\widehat{ln\phi}_{i,t}\right)\right]^{\zeta_2} \left\{\gamma\left[exp\left(\mu\right) - 1\right] + \frac{\nu}{2}[exp(\mu) - 1]^2\right\} \end{pmatrix} \left[\exp(\overline{ln_{\phi}}_t)\right]^{\frac{\alpha_\phi}{\frac{\sigma}{\sigma-1} - \alpha_k}} \\
&= \begin{matrix} \left[\exp(\overline{ln_{\phi}}_t)\right]^{\frac{\alpha_\phi}{\frac{\sigma}{\sigma-1} - \alpha_k} - \zeta_1} \left[\exp(\frac{1}{1-\rho}(-\mathbb{E}\tilde{\mu} - \bar{\mu}))\right]^{\zeta_1} \tilde{\phi}_{i,t}^{\zeta_1} \times f \times \mathbb{1}\{\mu_{i,t} > 0\} + \\ \left[\exp(\overline{ln_{\phi}}_t)\right]^{\frac{\alpha_\phi}{\frac{\sigma}{\sigma-1} - \alpha_k} - \zeta_2} \left[\exp(\frac{1}{1-\rho}(-\mathbb{E}\tilde{\mu} - \bar{\mu}))\right]^{\zeta_2} \tilde{\phi}_{i,t}^{\zeta_2} \left\{\gamma\left[exp\left(\mu\right) - 1\right] + \frac{\nu}{2}[exp(\mu) - 1]^2\right\} \end{matrix}
\end{aligned}$$

As shown in the previous subsection, the detrended model is isomorphic to the baseline model. Moreover, when $\eta = 0$, the model does not depend on the functional form of the collateral value of IP. Hence, we can calibrate the parameters such that the detrended optimal solution matches the baseline model solution before the policy intervention (i.e., when $\eta = 0$) in the main text. We then identify η using the same moment condition as in the baseline: the change in the share of firms engaging in R&D following the policy intervention. We simulate the transition path to match the observed 2.45 percentage point increase. This procedure yields $\eta_1 = 0.301$.

We use the estimated model to assess the nationwide implementation of the patent pledging policy. We analyze two outcomes:

1. **Static Gains**: We compare the ratio of actual to efficient TFP, $\ln(\mathrm{TFP}^{\mathrm{Act}}/\mathrm{TFP}^{\mathrm{Eff}})$, across BGPs. The policy improves allocative efficiency by 4.43%.

2. **Long-Run Growth**: The policy affects R&D investment $\tilde{\mu}_{i,t}$, which drives aggregate growth. The change in the growth rate is given by:

$$\Delta g = \vartheta \left(\int_i \tilde{\mu}_i^{\mathrm{post}} di - \int_i \tilde{\mu}_i^{\mathrm{pre}} di \right).$$

   Our results indicate that the policy increases the long-run growth rate of aggregate productivity by $\alpha_l \Delta g = 0.15$ percentage points.[58]

OA-H Extension: Exogenous Productivity Process

In this section, we specify a model without R&D investment, in which the productivity shock process is exogenous. Note that this restricted framework is similar to that of **?**, but the primary distinction is that firms in our model can leverage intellectual property (IP) to secure external funding. The model specification follows. The firm's optimization problem at the steady state is defined recursively as follows:

$$V(a, \phi) = \max_{\{c, a'\}} \left\{ \ln(c) + \beta \mathbb{E}[V(a', \phi') | \phi, \mu] \right\}$$

[58]Note that we assume the growth rate of the world economy increases by an equivalent amount, as a divergence in growth rates would violate our small open economy assumption.

subject to the budget constraint:

$$c + a' = \pi(a, \phi; w) + (1 + r)a,$$

where the profit function is given by:

$$\pi(a, \phi; w) = \max_{k,l} \left\{ (\phi^{\alpha_\phi} k^{\alpha_k} l^{\alpha_l})^{\frac{\sigma-1}{\sigma}} - wl - (r + \delta)k \right\},$$

subject to the collateral constraint:

$$k \leq \theta a + \eta \Psi(\phi).$$

The productivity process evolves according to:

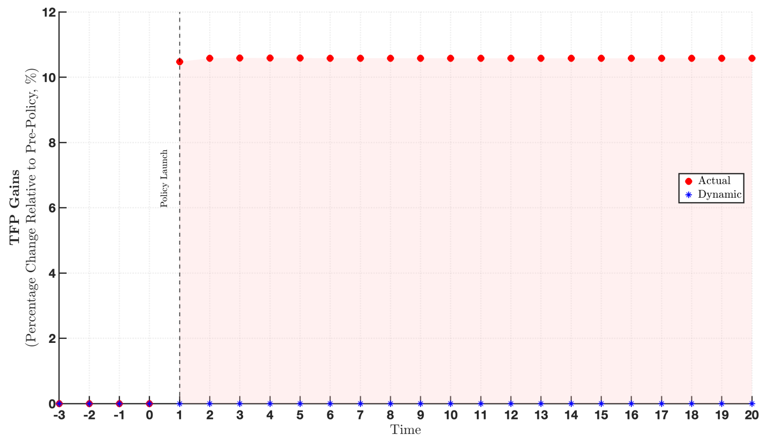
$$\ln(\phi') = \rho \ln(\phi) + \mu + \bar{\mu} + \sigma_\xi \xi',$$

where μ is taken as exogenous.

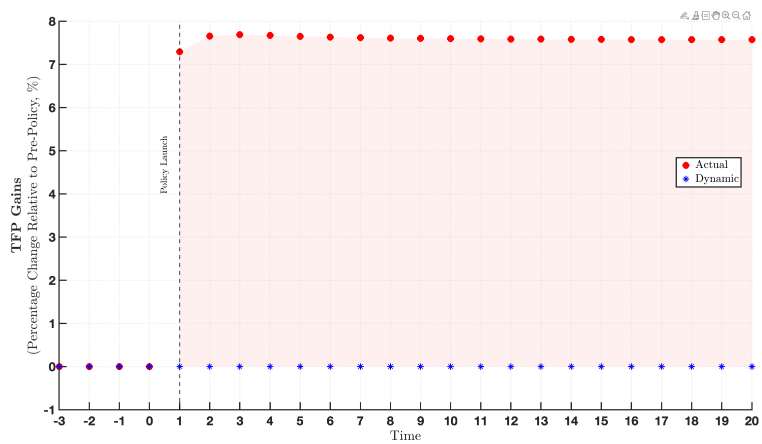
The equilibrium wage rate w clears the labor market such that total labor demand equals total supply $\bar{L}$.

OA-H.1 Calibration and Counterfactual Analysis

We do not re-estimate the model; instead, we retain the estimated parameters from the baseline and fix μ at its pre-policy mean. Using this calibration, we conduct counterfactual analyses by increasing η (or θ) to the levels used in the baseline. We then calculate the transition dynamics of actual TFP and the dynamic gains. These transition paths are shown in Figure OA-6a for the change in η and Figure OA-6b for the change in θ.



(a) Productivity Gains in General Equilibrium (Exogenous Shock Process), Policy Parameter $\eta = 0.1126$



(b) Productivity Gains in General Equilibrium (Exogenous Shock Process), Policy Parameter $\theta = 4.0995$